METHOD OF IMPULSE AND MOMENTUM

Until now, we have studied kinematics of particles, Newton's laws of motion and methods of work and energy. Newton's laws of motion describe relation between forces acting on a body at an instant and acceleration of the body at that instant. Therefore, it only helps us do analyze what is happening at an instant. The work kinetic energy theorem is obtained by integrating equation of motion ($\vec{F} = m\vec{a}$) over a path. Therefore, methods of work and energy help us to in exploring change in speed over a position interval. Now, we direct our attention on another principle – principle of impulse and momentum. It is obtained when equation of motion ($\vec{F} = m\vec{a}$) is integrated with respect to time. Therefore, this principle facilitates us with method to explore what is happening over a time interval.

Impulse of a Force

Net force applied on a rigid body changes momentum i.e. amount of motion of that body. A net force for a longer duration cause more change in momentum than the same force acting for shorter duration. Therefore duration in which a force acts on a body together with magnitude and direction of the force decide effect of the force on the change in momentum of the body.

Linear impulse or simply impulse of a force is defined as integral of the force with respect to time.

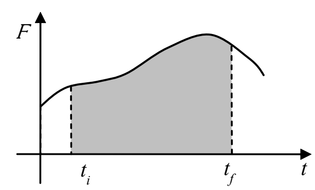
If a force $\vec{F}$ acts on a body, its impulse in a time interval from t_i to t_f is given by the following equation.

$$\vec{I}_{mp} = \int_{t_i}^{t_f} \vec{F}dt$$

If the force is constant, its impulse equals to product of the force vector $\vec{F}$ and time interval Δt.

$$\vec{I}_{mp} = \vec{F}(\Delta t)$$

For one-dimensional force, impulse equals to area between force-time graph and the time axis. In the given figure is shown how a force F along x-axis varies with time t. Impulse of this force in time interval t_i to t_f equals to area of the shaded portion.



If several forces $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$ ..... $\vec{F}_n$ act on a body in a time interval, the total impulse $\vec{I}_{mp}$ of all these forces equals to impulse of the net force.

$$\vec{I}_{mp} = \int_{t_i}^{t_f} \vec{F}_1 dt + \int_{t_i}^{t_f} \vec{F}_2 dt + \ldots\ldots\ldots\ldots + \int_{t_i}^{t_f} \vec{F}_n dt = \int_{t_i}^{t_f} \left(\vec{F}_1 + \vec{F}_2 + \ldots\ldots\ldots\ldots + \vec{F}_n\right) dt$$

Impulse is measured in newton-second.

Dimensions of impulse are MLT^{-1}

Example

Calculate impulse of force $\vec{F} = \left(3t^2\tilde{i} - (2t-1)\tilde{j} + 2\tilde{k}\right)$ N over the time interval from t = 1 s to t = 3 s.

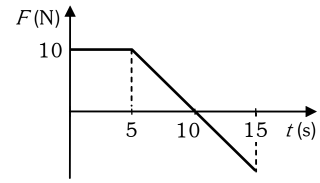
Solution.

$$\vec{I}_{mp} = \int_{t_i}^{t_f} \vec{F}dt \rightarrow \quad \vec{I}_{mp} = \int_1^3 \left(3t^2\tilde{i} - (2t-1)\tilde{j} + 2\tilde{k}\right) dt = \left[t^3\tilde{i} - \left(t^2 - t\right)\tilde{j} + 2t\tilde{k}\right]_1^3 = \tilde{i}\left[t^3\right]_1^3 - \tilde{j}\left[\left(t^2 - t\right)\right]_1^3 + \tilde{k}\left[2t\right]_1^3$$

$$= \left(26\tilde{i} - 6\tilde{j} + 4\tilde{k}\right) \text{ N-s}$$

Example

A one-dimensional force F varies with time according to the given graph. Calculate impulse of the force in following time intervals.



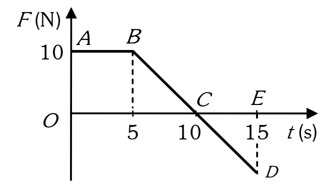
(a) From t = 0 s to t = 10 s.

(b) From t = 10 s to t = 15 s.

(c) From t = 0 s to t = 15 s.

Solution.

For one-dimensional force, impulse equals to area between force-time graph and the time axis.



(a) $I_{0\to 10}$ = Area of trapazium OABC = 75 N-s

(b) $I_{10\to 15}$ = – Area of triangle CDE = – 25 N-s

(c) $I_{0\to 15}$ = Area of trapazium OABC – Area of triangle CDE = 50 N-s

Impulse Momentum Principle

Consider body of mass m in translational motion. When it is moving with velocity $\vec{v}$, net external force acting on it is $\vec{F}$. Equation of motion as suggested by Newton's second law can be written in the form

$$\vec{F}dt = d(m\vec{v})$$

If the force acts during time interval from t_i to t_f and velocity of the body changes from $\vec{v}_i$ to $\vec{v}_f$, integrating the above equation with time over the interval from t_i to t_f, we have

$$\int_{t_i}^{t_f} \vec{F}dt = m\vec{v}_f - m\vec{v}_i$$

Here left hand side of the above equation is impulse $\vec{I}_{mp}$ of the net force $\vec{F}$ in time interval from t_i to t_f, and quantities $m\vec{v}_i$ and $m\vec{v}_f$ on the right hand side are linear momenta of the particle at instants t_i and t_f. If we denote them by symbols $\vec{p}_i$ and $\vec{p}_f$, the above equation can be written as

$$\vec{I}_{mp} = \vec{p}_f - \vec{p}_i$$

The idea expressed by the above equation is known as *impulse momentum principle*. It states that change in the momentum of a body in a time interval equals to the impulse of the net force acting on the body during the concerned time interval.

For the ease of application to physical situations the above equation is rearranged as

$$\vec{p}_i + \vec{I}_{mp} = \vec{p}_f$$

This equation states that impulse of a force during a time interval when added to momentum of a body at the beginning of an interval of time we get momentum of the body at the end of the interval concerned.

Since impulse and momentum both are vector quantities, the impulse momentum theorem can be expressed by there scalar equations making use of Cartesian components.

$$p_{1x} + \sum I_{mp,x} = p_{2x}$$

$$p_{1y} + \sum I_{mp,y} = p_{2y}$$

$$p_{1z} + \sum \mathrm{I}_{mp,z} = p_{2z}$$

The impulse momentum principle is deduced here for a single body moving relative to an inertial frame, therefore impulses of only physical forces are considered. If we are using a non-inertial reference frame, impulse of corresponding pseudo force must also be considered in addition to impulse of the physical forces.

How to apply Impulse Momentum Principle

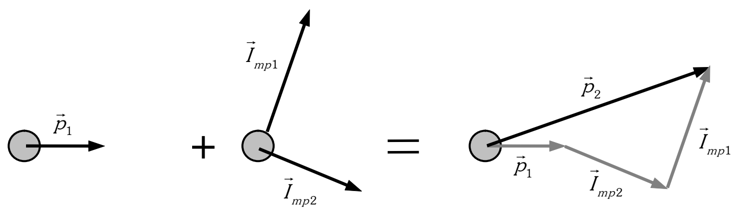
The impulse momentum principle is deduced here for a single body, therefore it is recommended at present to use it for a single body. To use this principle the following steps should be followed.

(i) Identify the initial and final positions as position 1 and 2 and show momenta $\vec{p}_1$ and $\vec{p}_2$ of the body at these instants.

(ii) Show impulse of each force acting on the body at an instant between positions 1 and 2.

(iii) Use the impulse obtained in step (ii) and momenta obtained in step (i) into equation $\vec{p}_i + \vec{I}_{mp} = \vec{p}_f$.

Consider a particle moving with momentum $\vec{p}_1$ in beginning. It is acted upon by two forces, whose impulses in a time interval are $\vec{I}_{mp1}$ and $\vec{I}_{mp2}$. As a result, at the end of the time interval, momentum of the particle becomes $\vec{p}_2$. This physical situation is shown in the following diagram. Such a diagram is known as *impulse momentum diagram*.



Example

A particle of mass 2 kg is moving with velocity $\vec{v}_o = (2\hat{i} - 3\hat{j})$ m/s in free space. Find its velocity 3 s after a constant force $\vec{F} = (3\hat{i} + 4\hat{j})$ N starts acting on it.

Solution.

$\vec{p}_f = \vec{p}_i + \vec{I}_{mp} \rightarrow \quad m\vec{v}_f = m\vec{v}_o + \vec{F}\Delta t$

Substituting given values, we have

$$2\vec{v}_f = 2(2\hat{i} - 3\hat{j}) + (3\hat{i} + 4\hat{j}) \times 3 = 13\hat{i} + 6\hat{j}$$

$$\vec{v}_f = (6.5\hat{i} + 3\hat{j}) \text{ m/s}$$

Example

A particle of mass 2 kg is moving in free space with velocity $\vec{v}_o = (2\hat{i} - 3\hat{j} + \hat{k})$ m/s is acted upon by force $\vec{F} = (2\hat{i} + \hat{j} - 2\hat{k})$ N. Find velocity vector of the particle 3 s after the force starts acting.

Solution.

$\vec{p}_f = \vec{p}_i + \vec{I}_{mp} \rightarrow \quad m\vec{v}_f = m\vec{v}_o + \vec{F}\Delta t$

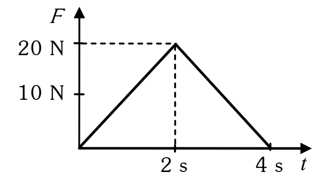
Substituting given values, we have

$$2\vec{v}_f = 2(2\hat{i} - 3\hat{j} + \hat{k}) + (2\hat{i} + \hat{j} - 2\hat{k}) \times 3 = 10\hat{i} - 3\hat{j} - 4\hat{k}$$

$$\vec{v}_f = (5\hat{i} - 1.5\hat{j} - 2\hat{k}) \text{ m/s}$$

Example

A box of mass m = 2 kg resting on a frictionless horizontal ground is acted upon by a horizontal force F, which varies as shown. Find speed of the particle when the force ceases to act.



Solution.

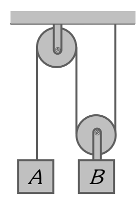
$$\vec{p}_f = \vec{p}_i + \vec{I}_{mp} \rightarrow \qquad m\vec{v}_f = m\vec{v}_i + \int_{t_i}^{t_f} \vec{F}dt$$

$$2v = 2 \times 0 + \tfrac{1}{2} \times 20 \times 4$$

$$v = 20 \text{ m/s}$$

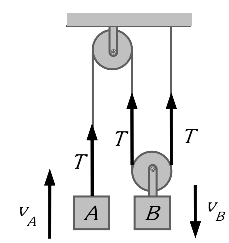
Example

Two boxes A and B of masses m and M interconnected by an ideal rope and ideal pulleys are held at rest as shown. When it is released, box B accelerates downwards. Find velocities of box A and B as function of time t after system has been released.



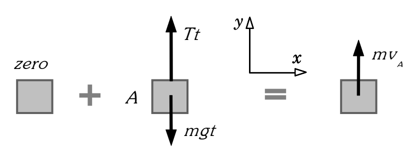
Solution.

We first explore relation between accelerations a_A and a_B of the boxes A and B, which can be written either by using constrained relation or method of virtual work or by inspection.



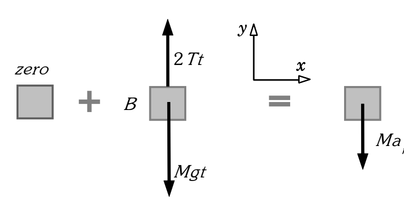
$$v_A = 2v_B \qquad \text{...(i)}$$

Applying impulse momentum principle to box A



$$p_{2y} = p_{1y} + \sum I_{mp,y} \rightarrow \qquad Mv_A = 0 + Tt - mgt \quad \text{...(ii)}$$

Applying impulse momentum principle to box B



$$p_{2y} = p_{1y} + \sum I_{mp,y} \rightarrow \qquad mv_B = 0 + Mgt - 2Tt \quad \text{...(iii)}$$

From equations (i), (ii) and (iii), we have

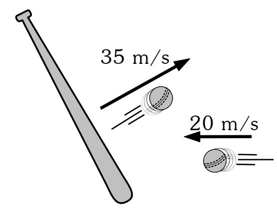
$$v_A = 2\left(\frac{M-2m}{M+4m}\right)gt \text{ and } v_B = \left(\frac{M-2m}{M+4m}\right)gt$$

Impulsive Motion

Sometimes a very large force acts for a very short time interval on a particle and produces finite change in momentum. Such a force is known as impulsive force and the resulting motion as impulsive motion. When a batsman hits a ball by bat, the contact between the ball and the bat lasts for a very small duration Δt, but the average value of the force F exerted by the bat on the ball is very large, and the resulting impulse $F\Delta t$ is large enough to change momentum of the ball.

During an impulsive motion, some other forces of magnitudes very small in comparison to that of an impulsive force may also act. Due to negligible time interval of the impulsive motion, impulse of these forces becomes negligible. These forces are known as non-impulsive forces. Effect of non-impulsive forces during an impulsive motion is so small that they are neglected in analyzing impulsive motion of infinitely small duration. Non-impulsive forces are of finite magnitude and include weight of a body, spring force or any other force of finite magnitude. When duration of the impulsive motion is specified, care has to be taken in neglecting any of the non-impulsive force. In analyzing motion of the ball for very small contact duration (usually in mili-seconds), impulse of the weight of the ball has to be neglected. Unknown reaction forces may be impulsive or non-impulsive; their impulse must therefore be included.

Example



A 100 gm ball moving horizontally with 20 m/s is struck by a bat, as a result it starts moving with a speed of 35 m/s at an angle of 37 above the horizontal in the same vertical plane as shown in the figure.

(a) Find the average force exerted by the bat if duration of impact is 0.30 s.

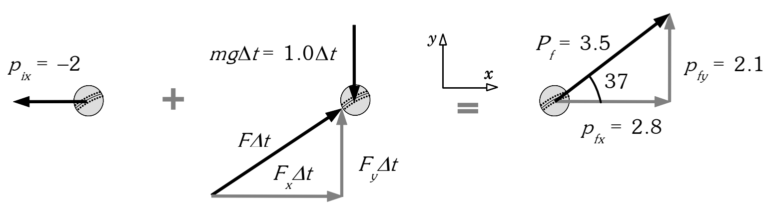
(b) Find the average force exerted by the bat if duration of impact is 0.03 s.

(c) Find the average force exerted by the bat if duration of impact is 0.003 s.

(d) What do you conclude for impulse of weight of the ball as duration of contact decreases?

Solution.

The impulse momentum diagram of the ball is shown in the figure below. Here F, mg, and Δt represent the average value of the force exerted by the bat, weight of the ball and the time interval.



Applying principle of impulse and momentum in x- direction, we have

$$p_{fx} = p_{ix} + \sum I_{mp,x} \rightarrow \qquad -2 + F_x\Delta t = 2.8$$

$$F_x = \frac{4.8}{\Delta t}\ \text{N} \qquad \ldots(i)$$

Applying principle of impulse and momentum in y- direction, we have

$$p_{fy} = p_{iy} + \sum I_{mp,y} \rightarrow \qquad 0.0 + F_y\Delta t - 1.0\Delta t = 2.1$$

$$F_y = \left(\frac{2.1}{\Delta t} + 1.0\right)\ \text{N} \qquad \ldots(ii)$$

(a) Substituting Δt = 0.30 s, in equations (i) and (ii), we find $\vec{F} = 16\hat{i} + 8\hat{j}$ N

(b) Substituting Δt = 0.03 s, in equations (i) and (ii), we find $\vec{F} = 160\hat{i} + 71\hat{j}$ N

(c) Substituting Δt = 0.003 s, in equations 1 and 2, we find $\vec{F} = 1600\hat{i} + 701\hat{j}$ N

(d) It is clear from the above results that as the duration of contact between the ball and the bat decreases, effect of the weight of the ball also decreases as compared with that of the force of the bat and for sufficiently short time interval, it can be neglected.

Momentum and Kinetic Energy

A moving particle possesses momentum as well as kinetic energy. If a particle of mass m is moving with velocity v, magnitude of its momentum p and its kinetic energy K bear the following relation.

$$K = \frac{p^2}{2m} = \frac{1}{2}pv$$

Example

An object is moving so that its kinetic energy is 150 J and the magnitude of its momentum is 30.0 kg-m/s. With what velocity is it traveling?

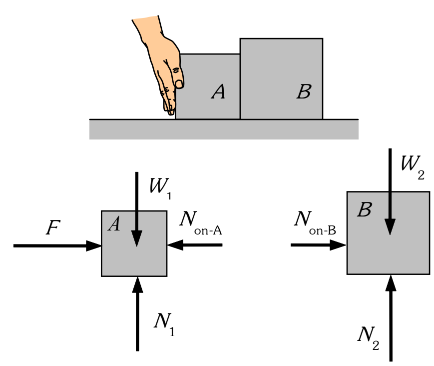
Solution.

$$K = \frac{p^2}{2m} = \frac{1}{2}pv \rightarrow v = \frac{2 \times 150}{30.0} = 10.0 \text{ m/s}$$

Internal and external Forces and System of interacting Particles

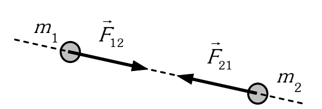
Bodies applying forces on each other are known as interacting bodies. If we consider them as a system, the forces, which they apply on each other, are known as internal forces and all other forces applied on them by bodies not included in the system are known as external forces.

Consider two blocks A and B placed on a frictionless horizontal floor. Their weights W_1 and W_2 are counterbalanced by normal reactions N_1 and N_2 on each of them from the floor. Push F by the hand is applied on A. The forces of normal reaction $N_{on\text{-}A}$ and $N_{on\text{-}B}$ constitute Newton's third law action-reaction pair, therefore are equal in magnitude and opposite in direction. Among these forces weights W_1 and W_2 applied by the earth, normal reactions N_1 and N_2 applied by the ground and the push F applied by the hand are external forces and normal reactions $N_{on\text{-}A}$ and $N_{on\text{-}B}$ are internal forces.

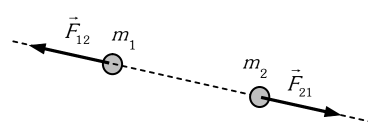


If the blocks are connected by a spring and the block A is either pushed or pulled, the forces W_1, W_2, N_1 and N_2 still remain external forces for the two block system and the forces, which the spring applies on each other are the internal forces. Here force of gravitational interaction between them being negligible has been neglected.

We can conceive a general model of two interacting particles. In the figure is shown a system of two particles of masses m_1 and m_2. Particle m_1 attracts m_2 with a force $\vec{F}_{12}$ and m_2 attracts (or pulls) m_1 with a force $\vec{F}_{21}$. These forces $\vec{F}_{12}$ and $\vec{F}_{21}$ are the internal forces of this two-particle system and are equal in magnitude and opposite in directions. Instead of attraction may repeal each other. Such a system of two particles repealing each other is also shown.

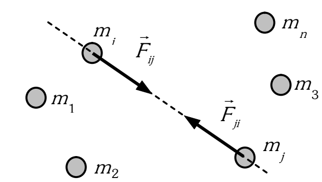


Particles attracting each other



Particles repealing each other

In similar way we may conceive a model of a system of n interacting particles having masses $m_1, m_2, \ldots m_i \ldots m_j \ldots$..and m_n respectively. The forces of interaction $\vec{F}_{ij}$ and $\vec{F}_{ji}$ between m_i and m_j are shown in the figure. Similar to these other particles may also interact with each other. These forces of mutual interaction between the particles are internal forces of the system. Any of the two interacting particles always apply equal and opposite forces on each other. Here fore simplicity only the forces $\vec{F}_{ij}$ and $\vec{F}_{ji}$ are shown.



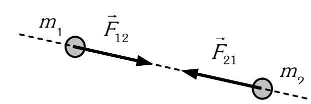
System of n interacting particles.

Principle of Conservation of linear momentum

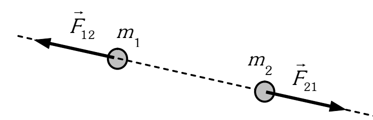
The principle of conservation of linear momentum or simply conservation of momentum for two or more interacting bodies is one of guiding principles of the classical as well as the modern physics.

To understand this principle, we first discuss a system of two interacting bodies, and then extend the ideas developed to a system consisting of many interacting bodies.

Consider a system of two particles of mass m_1 and m_2. Particle m_1 attracts m_2 with a force $\vec{F}_{12}$ and m_2 attracts m_1 with a force $\vec{F}_{21}$. These forces have equal magnitudes and opposite directions as shown in the figure. If the bodies are let free i.e. without any external force acting on any of them, each of them move and gain momentum equal to the impulse of the force of interaction. Since equal and opposite interaction forces act on both of them for the same time interval, the momenta gained by them are equal in magnitude and opposite in direction resulting no change in total momentum of the system.



Particles attracting each other

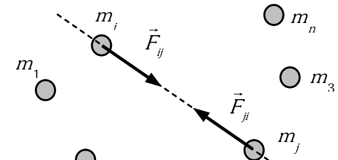


Particles repealing each other

However, if an external force acts on any one of them or different forces with a nonzero resultant act on both of them, the total momentum of the bodies will certainly change. If the system undergoes an impulsive motion, total momentum will change only under the action of external impulsive force or forces. Internal impulsive forces also exist in pairs of equal and opposite forces and cannot change the total momentum of the system. Non-impulsive forces if act cannot change momentum of the system by appreciable amount. For example, gravity is a non-impulsive force, therefore in the process of collision between two bodies near the earth the total momentum remains conserved.

The total momentum of a system of two interacting bodies remains unchanged under the action of the forces of interaction between them. It can change only if a net impulse of external force is applied.

In similar way we may conceive a model of a system of n interacting particles having masses $m_1, m_2, \ldots m_i \ldots m_j \ldots$..and m_n respectively. The forces of interaction $\vec{F}_{ij}$ and $\vec{F}_{ji}$ between m_i and m_j are shown in the figure. Since internal forces exist in pairs of equal and opposite forces, in any time interval of concern each of them have a finite impulse but their total impulse is zero. Thus if the system is let free, in any time interval momentum of every individual particle changes but the total momentum of the system remains constant.



System of n interacting particles.

It can change only if external forces are applied to some or all the particles. Under the action of external forces, the change in total momentum of the system will be equal to the net impulse of all the external forces.

Thus, total momentum of a system of particles cannot change under the action of internal forces and if net impulse of the external forces in a time interval is zero, the total momentum of the system in that time interval will remain conserved.

$$\sum \vec{p}_{initial} = \sum \vec{p}_{final}$$

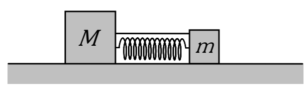
The above statement is known as the principle of conservation of momentum. It is applicable only when the net impulse of all the external forces acting on a system of particles becomes zero in a finite time interval. It happens in the following conditions.

- When no external force acts on any of the particles or bodies.
- When resultant of all the external forces acting on all the particles or bodies is zero.
- In impulsive motion, where time interval is negligibly small, the direction in which no impulsive forces act, total component of momentum in that direction remains conserved.

Since force, impulse and momentum are vectors, component of momentum of a system in a particular direction is conserved, if net impulse of all external forces in that direction vanishes.

Example

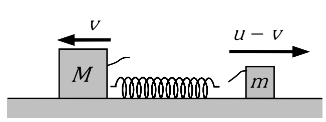
Two blocks of masses m and M are held against a compressed spring on a frictionless horizontal floor with the help of a light thread. When the thread is cut, the smaller block leaves the spring with a velocity u relative to the larger block. Find the recoil velocity of the larger block.



Solution.

When the thread is cut, the spring pushes both the block, and impart them momentum. The forces applied by the spring on both the block are internal forces of the two-block system. External forces acting on the system are weights and normal reactions on the blocks from the floor. These external forces have zero net resultant of the system. In addition to this fact no external force acts on the system in horizontal direction, therefore, horizontal component of the total momentum of the system remains conserved.

Velocities of both the objects relative to the ground (inertial frame) are shown in the adjoining figure.



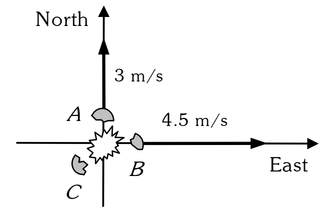
Since before the thread is cut system was at rest, its total momentum was zero. Principle of conservation of momentum for the horizontal direction yields

$$\sum_{i=1}^{n} p_{horizontal} = 0 \rightarrow \qquad -Mv + m(u - v) = 0$$

$$v = \frac{mu}{M+m}$$

Example

A shell fired vertically up, when reaches its highest point, explodes into three fragments A, B and C of masses m_A = 4 kg, m_B = 2 kg and m_C = 3kg. Immediately after the explosion, A is observed moving with velocity v_A = 3 m/s towards north and B with a velocity v_B = 4.5 m/s towards east as shown in the figure. Find the velocity v_C of the piece C.



Solution.

Explosion takes negligible duration; therefore, impulse of gravity, which is a finite external force, can be neglected. The pieces fly off acquiring above-mentioned velocities due to internal forces developed due to expanding gases produced during the explosion. The forces applied by the expanding gases are internal forces; hence, momentum of the system of the three pieces remains conserved during the explosion and total momentum before and after the explosion are equal.

Assuming the east as positive x-direction and the north as positive y-direction, the momentum vectors $\vec{p}_A$ and $\vec{p}_B$ of pieces A and B become

$$\vec{p}_A = m_A v_A \tilde{j} = 12 \text{ kg-m/s} \quad \text{and} \quad \vec{p}_B = m_B v_B \tilde{i} = 9 \text{ kg-m/s}$$

Before the explosion, momentum of the shell was zero, therefore from the principle of conservation of momentum, the total momentum of the fragments also remains zero.

$$\vec{p}_A + \vec{p}_B + \vec{p}_C = \vec{0} \rightarrow \qquad \vec{p}_C = -(9\tilde{i} + 12\tilde{j})$$

From the above equation, velocity of the piece C is

$$\vec{v}_C = \frac{\vec{p}_C}{m_C} = -(3\tilde{i} + 4\tilde{j}) = 5 \text{ m/s, } 53 \text{ south of west.}$$

Example

In free space, three identical particles moving with velocities $v_o\tilde{i}$, $-3v_o\tilde{j}$ and $5v_o\tilde{k}$ collide successively with each other to form a single particle. Find velocity vector of the particle formed.

Solution.

Let m be the mass of a single particle before any of the collisions. The mass of particle formed after collisions must be 3m. In free space, no external forces act on any of the particles, their total momentum remains conserved.

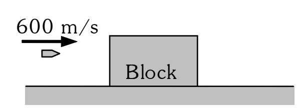
Applying principle of conservation of momentum, we have

$$\sum \vec{p}_{initial} = \sum \vec{p}_{final} \rightarrow \qquad m v_o \tilde{i} - 3 m v_o \tilde{j} + 5 m v_o \tilde{k} = 3 m \vec{v}$$

$$\vec{v} = \tfrac{1}{3} v_o \left(\tilde{i} - 3\tilde{j} + 5\tilde{k}\right) \text{ m/s}$$

Example

A bullet of mass 50 g moving with velocity 600 m/s hits a block of mass 1.0 kg placed on a rough horizontal ground and comes out of the block with a velocity of 400 m/s. The coefficient of friction between the block and the ground is 0.25. Neglect loss of mass of the block as the bullet pierces through it.



(a) In spite of the fact that friction acts as an external force, can you apply principle of conservation of momentum during interaction of the bullet with the block?

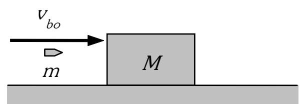
(b) Find velocity of the block immediately after the bullet pierces through it.

(c) Find the distance the block will travel before it stops.

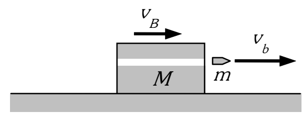
Solution.

(a) There is no net external force in the vertical direction and in the horizontal direction, only external force friction is non-impulsive, therefore momentum of the bullet-block system during their interaction remains conserved.

(b) Let us denote velocities of the bullet before it hits the block and immediately after it pierces through the block by v_{bo} and v_b, velocity of the block immediately after the bullet pierces through it is v_B and masses of the bullet and the block by m and M respectively. These are shown in the adjacent figure.



Immediately before the bullet hits the block



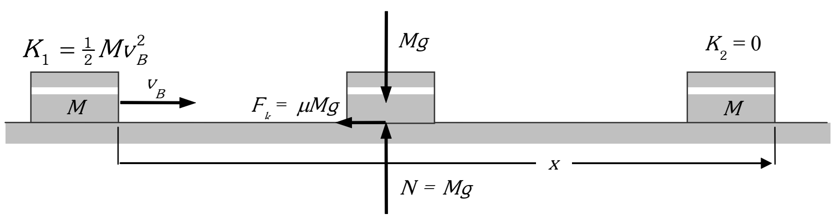
Immediately after the bullet pierces the block

Applying principle of conservation of momentum for horizontal component, we have

$$mv_{bo} = mv_b + Mv_B \rightarrow \qquad v_B = \frac{m(v_{bo} - v_b)}{M}$$

Substituting the given values, we have v_B = 10 m/s

(c) To calculate distance traveled by the block before it stops, work kinetic energy theorem has to be applied.



During sliding of the box on the ground only the force of kinetic friction does work.

$$W_{1\rightarrow 2} = K_2 - K_1 \rightarrow \qquad -\mu Mgx = 0 - \tfrac{1}{2}Mv_B^2$$

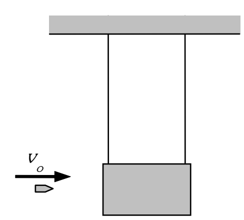
$$x = \frac{v_B^2}{2\mu g}$$

Substituting given values, we have x = 20 m

Example

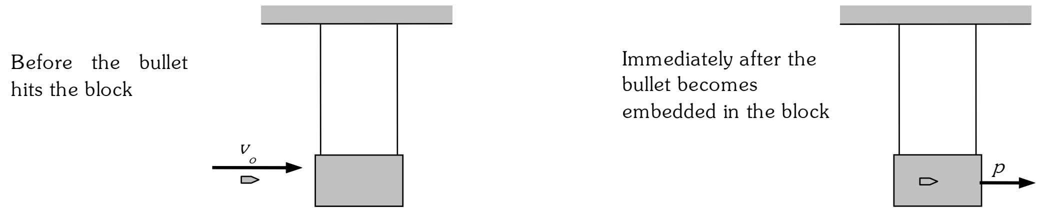
Ballistic Pendulum : A ballistic pendulum is used to measure speed of bullets. It consists of a wooden block suspended from fixed support.

A wooden block of mass M is suspended with the help of two threads to prevent rotation while swinging. A bullet of mass m moving horizontally with velocity v_o hits the block and becomes embedded in the block. Receiving momentum from the bullet, the bullet-block system swings to a height h. Find expression for speed of the bullet in terms of given quantities.



Solution.

When the bullet hits the block, in a negligible time interval, it becomes embedded in the block and the bullet-block system starts moving with horizontally. During this process, net force acting on the bullet-block system in vertical direction is zero and no force acts in the horizontal direction. Therefore, momentum of the bullet-block system remains conserved.



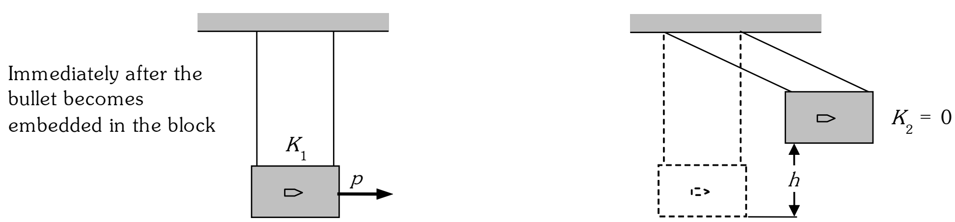
Let us denote momentum of the bullet-block system immediately after the bullet becomes embedded in the block by p and apply principle of conservation of momentum to the system for horizontal component of momentum.

$$p = mv_o$$

Using equation $K = p^2 / 2m$, we can find kinetic energy K_1 of the bullet-block system immediately after the bullet becomes embedded in the block.

$$K_1 = \frac{(mv_o)^2}{2(M+m)}$$

During swing, only gravity does work on the bullet-block system. Applying work-kinetic energy theorem during swing of the bullet-block system, we have



$$W_{1\to2} = K_2 - K_1 \rightarrow \qquad -(M+m)gh = 0 - \frac{(mv_o)^2}{2(M+m)}$$

Rearranging terms, we have

$$v_o = \frac{(M+m)}{m}\sqrt{2gh}$$

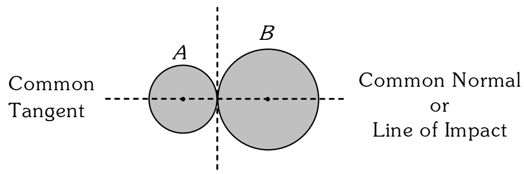
Impact between two bodies

Impact or collision is interaction of very small duration between two bodies in which the bodies apply relatively large forces on each other.

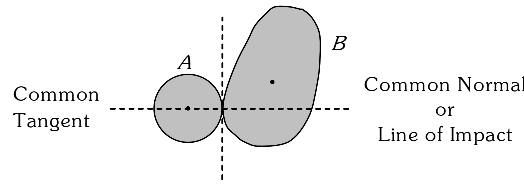
Interaction forces during an impact are created due to either direct contact or strong repulsive force fields or some connecting links. These forces are so large as compared to other external forces acting on either of the bodies that the effects of later can be neglected. The duration of the interaction is short enough as compared to the time scale of interest as to permit us only to consider the states of motion just before and after the event and not during the impact. Duration of an impact ranges from 10^{-23} s for impacts between elementary particles to millions of years for impacts between galaxies. The impacts we observe in our everyday life like that between two balls last from 10^{-3} s to few seconds.

Central and Eccentric Impact

The common normal at the point of contact between the bodies is known as line of impact. If mass centers of the both the colliding bodies are located on the line of impact, the impact is called central impact and if mass centers of both or any one of the colliding bodies are not on the line of impact, the impact is called eccentric impact.



Central Impact

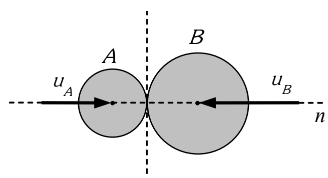


Eccentric Impact

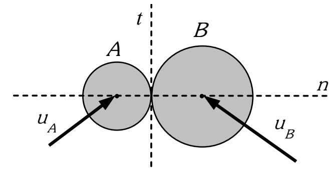
Central impact does not produce any rotation in either of the bodies whereas eccentric impact causes the body whose mass center is not on the line of impact to rotate. Therefore, at present we will discuss only central impact and postpone analysis of eccentric impact to cover after studying rotation motion.

Head–on (Direct) and Oblique Central Impact

If velocities vectors of the colliding bodies are directed along the line of impact, the impact is called a direct or head-on impact; and if velocity vectors of both or of any one of the bodies are not along the line of impact, the impact is called an oblique impact.



Central Impact

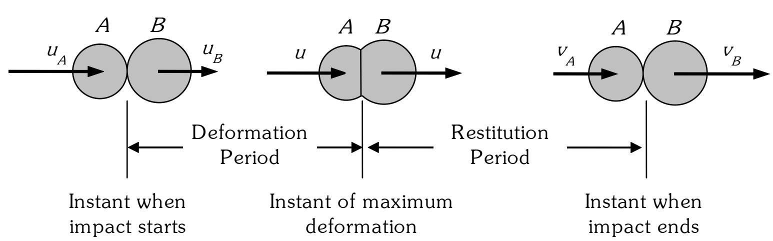


Oblique Impact

In this chapter, we discuss only central impact, therefore the term central we usually not use and to these impacts, we call simply head-on and oblique impacts. Furthermore, use of the line of impact and the common tangent is so frequent in analysis of these impacts that we call them simply *t*-axis and *n*-axis.

Head–on (Direct) Central Impact

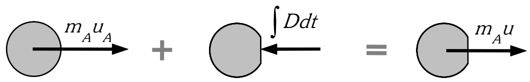
To understand what happens in a head-on impact let us consider two balls *A* and *B* of masses m_A and m_B moving with velocities u_A and u_B in the same direction as shown. Velocity u_A is larger than u_B so the ball A hits the ball *B*. During impact, both the bodies push each other and first they get deformed till the deformation reaches a maximum value and then they tries to regain their original shape due to elastic behaviors of the materials forming the balls.



The time interval when deformation takes place is called the deformation period and the time interval in which the ball try to regain their original shapes is called the restitution period. Due to push applied by the balls on each other during period of deformation speed of the ball A decreases and that of the ball *B* increases and at the end of the deformation period, when the deformation is maximum both the ball move with the same velocity say it is *u*. Thereafter, the balls will either move together with this velocity or follow the period of restitution. During the period of restitution due to push applied by the balls on each other, speed of the ball *A* decrease further and that of ball B increase further till they separate from each other. Let us denote velocities of the balls *A* and *B* after the impact by v_A and v_B respectively.

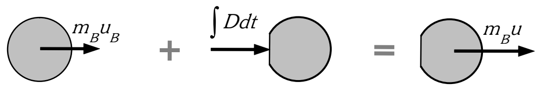
Equation of Impulse and Momentum during impact

Impulse momentum principle describes motion of ball A during deformation period.



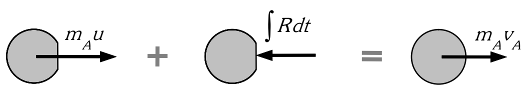
$$m_A u_A - \int Ddt = m_A u \qquad \ldots\text{(i)}$$

Impulse momentum principle describes motion of ball B during deformation period.



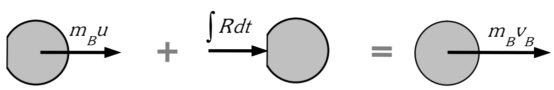
$$m_B u_B + \int Ddt = m_B u \qquad \ldots\text{(ii)}$$

Impulse momentum principle describes motion of ball A during restitution period.



$$m_A u - \int Rdt = m_A v_A \qquad \ldots\text{(iii)}$$

Impulse momentum principle describes motion of ball B during restitution period.



$$m_B u + \int Rdt = m_B v_B \qquad \ldots\text{(iv)}$$

Conservation of Momentum during impact

From equations, (i) and (ii) we have $\quad m_A u_A + m_B u_B = \left(m_A + m_B\right)u \quad \ldots$(v)

From equations, (iii) and (iv) we have $\quad \left(m_A + m_B\right)u = m_A v_A + m_B v_B \quad \ldots$(vi)

From equations, (v) and (vi) we obtain the following equation.

$$m_A v_A + m_B v_B = m_A u_A + m_B u_B \qquad \ldots\text{(vii)}$$

The above equation elucidates the principle of conservation of momentum.

Coefficient of Restitution

Usually the force D applied by the bodies A and B on each other during period differs from the force R applied by the bodies on each other during period of restitution. Therefore, it is not necessary that magnitude of impulse $\int Ddt$ of deformation equals to the magnitude of impulse $\int Rdt$ restitution.

The ratio of magnitudes of impulse of restitution to that of deformation is called the coefficient of restitution and is denoted by e.

$$e = \frac{\int Rdt}{\int Ddt} \qquad \ldots\text{(viii)}$$

Now from equations (i), (ii), (iii) and (iv), we have

$$e = \frac{v_B - v_A}{u_A - u_B} \qquad \ldots\text{(ix)}$$

Coefficient of restitution depend on various factors as elastic properties of materials forming the bodies, velocities of the contact points before impact, state of rotation of the bodies and temperature of the bodies. In general, its value ranges from zero to one but in collision where kinetic energy is generated its value may exceed one.

Depending on values of coefficient of restitution, two particular cases are of special interest.

Perfectly Plastic or Inelastic Impact	For these impacts e = 0, and bodies undergoing impact stick to each other after the impact.
Perfectly Elastic Impact	For these impacts e = 1.

Strategy to solve problems of head-on impact

Write momentum conservation equation

$$m_A v_A + m_B v_B = m_A u_A + m_B u_B \qquad ...(A)$$

Write rearranging terms of equation of coefficient of restitution

$$v_B - v_A = e(u_A - u_B) \qquad ...(B)$$

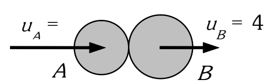
Use the above equations A and B.

Example

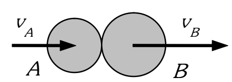
A ball of mass 2 kg moving with speed 5 m/s collides directly with another of mass 3 kg moving in the same direction with speed 4 m/s. The coefficient of restitution is 2/3. Find the velocities after collision.

Solution.

Denoting the first ball by A and the second ball by B velocities immediately before and after the impact are shown in the figure.



Immediately before impact starts



Immediately after impact ends

Applying principle of conservation of momentum, we have

$$m_B v_B + m_A v_A = m_A u_A + m_B u_B \rightarrow\ 3v_B + 2v_A = 2 \times 5 + 3 \times 4$$

$$3v_B + 2v_A = 22 \qquad ...(i)$$

Applying equation of coefficient of restitution, we have

$$v_B - v_A = e(u_A - u_B) \rightarrow \qquad v_B - v_A = \tfrac{2}{3}(5-4)$$

$$3v_B - 3v_A = 2 \qquad ...(ii)$$

From equation (i) and (ii), we have v_A = 4 m/s and v_B = 4.67 m/s **Ans.**

Example

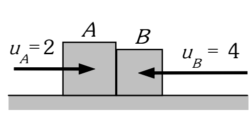
A block of mass 5 kg moves from left to right with a velocity of 2 m/s and collides with another block of mass 3 kg moving along the same line in the opposite direction with velocity 4 m/s.

(a) If the collision is perfectly elastic, determine velocities of both the blocks after their collision.

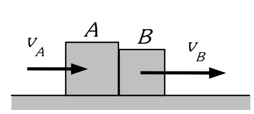
(b) If coefficient of restitution is 0.6, determine velocities of both the blocks after their collision.

Solution.

Denoting the first block by A and the second block by B velocities immediately before and after the impact are shown in the figure.



Immediately before impact starts



Immediately after impact ends

Applying principle of conservation of momentum, we have

$$m_B v_B + m_A v_A = m_A u_A + m_B u_B \rightarrow\ 3v_B + 5v_A = 5 \times 2 + 3 \times (-4)$$

$$3v_B + 5v_A = -2 \qquad ...(i)$$

Applying equation of coefficient of restitution, we have

$$v_B - v_A = e(u_A - u_B) \rightarrow \qquad v_B - v_A = e\{2-(-4)\}$$

$$v_B - v_A = 6e \qquad \text{...(ii)}$$

(a) For perfectly elastic impact $e = 1$. Using this value in equation (ii), we have

$$v_B - v_A = 6 \qquad \text{...(iia)}$$

Now from equation (i) and (iia), we obtain

$$v_A = -2.5 \text{ m/s and } v_B = 3.5 \text{ m/s}$$

(b) For value e = 0.6, equation 2 is modified as

$$v_B - v_A = 3.6 \qquad \text{(iib)}$$

Now from equation (i) and (iib), we obtain

$$v_A = -1.6 \text{ m/s and } v_B = 2.0 \text{ m/s}$$

Block A reverse back with speed 1.6 m/s and B also move in opposite direction to its original direction with speed 2.0 m/s.

Example

Two identical balls A and B moving with velocities u_A and u_B in the same direction collide. Coefficient of restitution is e.

(a) Deduce expression for velocities of the balls after the collision.

(b) If collision is perfectly elastic, what do you observe?

Solution.

Equation expressing momentum conservation is

$$v_A + v_B = u_A + u_B \qquad \text{...(A)}$$

Equation of coefficient of restitution is

$$v_B - v_A = eu_A - eu_B \qquad \text{...(B)}$$

(a) From the above two equations, velocities v_A and v_B are

$$v_A = \left(\frac{1-e}{2}\right)u_A + \left(\frac{1+e}{2}\right)u_B \qquad \text{...(i)}$$

$$v_B = \left(\frac{1+e}{2}\right)u_A + \left(\frac{1-e}{2}\right)u_B \qquad \text{...(ii)}$$

(b) For perfectly elastic impact $e = 1$, velocities v_A and v_B are

$$v_A = u_B \qquad \text{...(iii)}$$

$$v_B = u_A \qquad \text{...(iv)}$$

Identical bodies exchange their velocities after perfectly elastic impact.

Conservation of kinetic energy in perfectly elastic impact

For perfectly elastic impact equation for conservation of momentum and coefficient of restitution are

$$m_A v_A + m_B v_B = m_A u_A + m_B u_B \qquad \text{...(A)}$$

$$v_B - v_A = u_A - u_B \qquad \text{...(B)}$$

Rearranging the terms of the above equations, we have

$$m_A(v_A - u_A) = m_B(u_B - v_B)$$

$$u_A + v_A = v_B + u_B$$

Multiplying LHS of both the equations and RHS of both the equations, we have

$$m_A\left(v_A^2-u_A^2\right)=m_B\left(u_B^2-v_B^2\right)$$

Multiplying by ½ and rearranging term of the above equation, we have

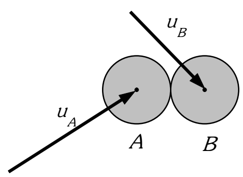
$$\tfrac{1}{2}m_Au_A^2+\tfrac{1}{2}m_Bu_B^2=\tfrac{1}{2}m_Av_A^2+\tfrac{1}{2}m_Bv_B^2$$

In perfectly elastic impact total kinetic energy of the colliding body before and after the impact are equal.

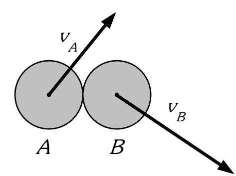
In inelastic impacts, there is always loss of kinetic energy.

Oblique Central Impact

In oblique central impact, velocity vectors of both or of any one of the bodies are not along the line of impact and mass center of bodies are on the line of impact. Due to impact speeds and direction of motion of both the balls change. In the given figure is shown two balls A and B of masses m_A and m_B moving with velocities u_A and u_B collide obliquely. After the collision let they move with velocities v_A and v_B as shown in the nest figure.

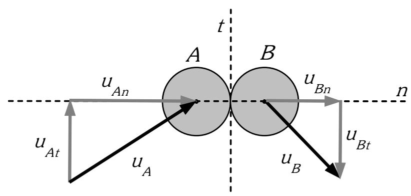


Immediately before Impact

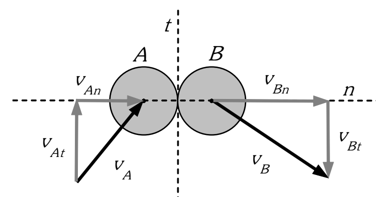


Immediately after Impact

To analyze the impact, we show components of velocities before and after the impact along the common tangent and the line of impact. These components are shown in the following figure.



Immediately before Impact



Immediately before Impact

Component along the t-axis

If surfaces of the bodies undergoing impact are smooth, they cannot apply any force on each other along the *t*-axis and component of momentum along the *t*-axis of each bodies, considered separately, is conserved. Hence, *t*-component of velocities of each of the bodies remains unchanged.

$$v_{At}=u_{At} \text{ and } v_{Bt}=u_{Bt} \quad \text{...(A)}$$

Component along the n-axis

For components of velocities along the n-axis, the impact can be treated same as head-on central impact.

The component along the n-axis of the total momentum of the two bodies is conserved

$$m_Bv_{Bn}+m_Av_{An}=m_Bu_{Bn}+m_Au_{An} \quad \text{...(B)}$$

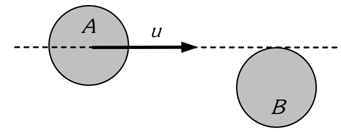
Concept of coefficient of restitution e is applicable only for the *n*-component velocities.

$$v_{Bn}-v_{An}=e\left(u_{An}-u_{Bn}\right) \quad \text{...(C)}$$

The above four independent equation can be used to analyze oblique central impact of two freely moving bodies.

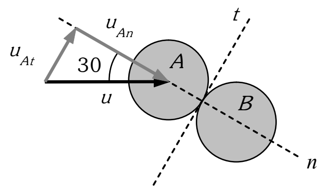
Example

A disk sliding with velocity u on a smooth horizontal plane strikes another identical disk kept at rest as shown in the figure. If the impact between the disks is perfectly elastic impact, find velocities of the disks after the impact.

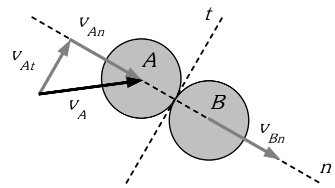


Solution.

(a) We first show velocity components along the t and the n-axis immediately before and after the impact. angle that the line of impact makes with velocity u is 30 .



Immediately before Impact



Immediately after Impact

Component along t-axis

Components of momentum along the *t*-axis of each disk, considered separately, is conserved. Hence, *t*-component of velocities of each of the bodies remains unchanged.

$$v_{At} = u_{At} = \frac{u}{2} \text{ and } v_{Bt} = u_{Bt} = 0 \qquad \text{...(i)}$$

Component along n-axis

The component along the *n*-axis of the total momentum of the two bodies is conserved

$$m_B v_{Bn} + m_A v_{An} = m_B u_{Bn} + m_A u_{An} \rightarrow \; m v_{Bn} + m v_{An} = m \times 0 + m\frac{u\sqrt{3}}{2}$$

$$v_{Bn} + v_{An} = \frac{u\sqrt{3}}{2} \qquad \text{...(ii)}$$

Concept of coefficient of restitution *e* is applicable only for the *n*-component velocities.

$$v_{Bn} - v_{An} = e\left(u_{An} - u_{Bn}\right) \rightarrow \quad v_{Bn} - v_{An} = \frac{u\sqrt{3}}{2} \qquad \text{...(iii)}$$

From equations (ii) and (iii), we have $v_{An} = 0$ and $v_{Bn} = \frac{u\sqrt{3}}{2}$ ...(iv)

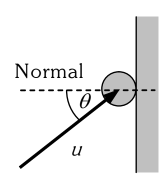
From equations (i) and (iv) we can write velocities of both the disks.

Example

A ball collides with a frictionless wall with velocity u as shown in the figure. Coefficient of restitution for the impact is *e*.

(a) Find expression for the velocity of the ball immediately after the impact.

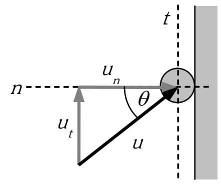
(b) If impact is perfectly elastic what do you observe?



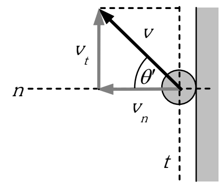
Solution.

(a) Let us consider the ball as the body *A* and the wall as the body *B*. Since the wall has infinitely large inertia (mass) as compared to the ball, the state of motion of the wall, remains unaltered during the impact i.e. the wall remain stationary.

Now we show velocities of the ball and its t and n-components immediately before and after the impact. For the purpose we have assumed velocity of the ball after the impact *v*.



Immediately before Impact



Immediately after Impact

Component along t-axis

Components of momentum along the *t*-axis of the ball is conserved. Hence, *t*-component of velocities of each of the bodies remains unchanged.

$$v_t = u_t = u\sin\theta \quad \ldots\text{(i)}$$

Component along n-axis

Concept of coefficient of restitution e is applicable only for the *n*-component velocities.

$$v_{Bn} - v_{An} = e\left(u_{An} - u_{Bn}\right) \rightarrow \qquad -v_n = eu_n$$

$$v_n = -eu\cos\theta \quad \ldots\text{(ii)}$$

From equations (i) and (ii), the *t* and *n*-components of velocity of the ball after the impact are

$$v_t = u\sin\theta \quad \text{and} \quad v_n = eu\sin\theta$$

(b) If the impact is perfectly elastic, we have $v_t = u\sin\theta$, $v_n = u\sin\theta$ and $\theta' = \theta$

The ball will rebound with the same speed making the same angle with the vertical at which it has collided. In other words, a perfectly elastic collision of a ball with a wall follows the same laws as light follows in reflection at a plane mirror.

Oblique Central Impact when one or both the colliding bodies are constrained in motion

In oblique collision, we have discussed how to analyze impact of bodies that were free to move before as well as after the impact. Now we will see what happens if one or both the bodies undergoing oblique impact are constrained in motion.

Component along the t-axis

If surfaces of the bodies undergoing impact are smooth, the *t*-component of the momentum of the body that is free to move before and after the impact remain conserved.

If both the bodies are constrained, the *t*-component of neither one remains conserved.

Momentum Conservation

We may find a direction in which no external force acts on both the bodies. The component of total momentum of both the bodies along this direction remains conserved.

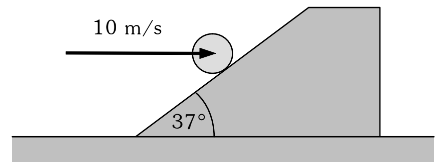
Coefficient of restitution

Concept of coefficient of restitution e is applicable only for the n-component velocities.

$$v_{Bn} - v_{An} = e\left(u_{An} - u_{Bn}\right)$$

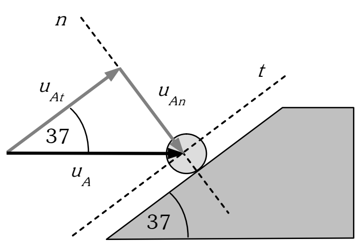
Example

A 250 g ball moving horizontally with velocity 10.0 m/s strikes inclined surface of a 720 g smooth wedge as shown in the figure. The wedge is placed at rest on a frictionless horizontal ground. If the coefficient of restitution is 0.8, calculate the velocity of the wedge after the impact.

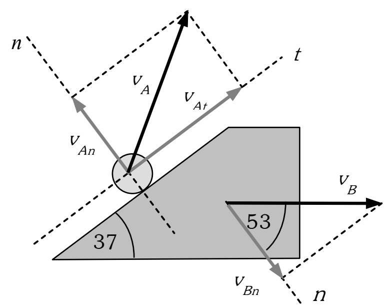


Solution.

Let us consider the ball as the body *A* and the wedge as the body *B*. After the impact, the ball bounces with velocity v_A and the wedge advances in horizontal direction with velocity v_B. These velocities and their *t* and *n*-components are immediately before and after the impact are shown in the following figures.



Immediately before Impact



Immediately after Impact

Component along t-axis

The ball is free to move before and after the impact, therefore its *t*-component of momentum conserved. Hence, *t*-component of velocities of the ball remains unchanged.

$$v_{At} = u_{At} = 10\cos 37° = 8 \text{ m/s} \qquad \text{...(i)}$$

Momentum Conservation

In the horizontal direction, there is no external force on both the bodies. Therefore horizontal component of total momentum of both the bodies remain conserved.

$$m_A u_A = m_A\left(v_{At}\cos 37° - v_{An}\cos 53°\right) + m_B v_B$$

$$0.25 \times 10 = 0.25\left(8 \times \frac{4}{5} - \frac{3v_{An}}{5}\right) + 0.72 v_B \qquad \text{...(ii)}$$

Coefficient of restitution

Concept of coefficient of restitution *e* is applicable only for the *n*-component velocities.

$$v_{Bn} - v_{An} = e\left(u_{An} - u_{Bn}\right) \rightarrow \frac{3v_B}{5} - v_{An} = 0.8(6-0) \qquad \text{...(iii)}$$

From equations (i), (ii) and (iii), we obtain v_B = 2.0 m/s

SYSTEM OF PARTICLES

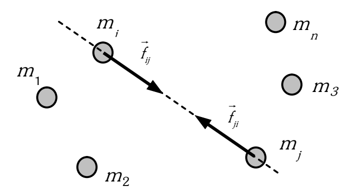
Study of kinematics enables us to explore nature of translation motion without any consideration to forces and energy responsible for the motion. Study of kinetics enables us to explore effects of forces and energy on motion. It includes Newton's laws of motion, methods of work and energy and methods of impulse and momentum. The methods of work and energy and methods of impulse and momentum are developed using equation $\vec{F} = m\vec{a}$ together with the *methods of kinematics*. The advantage of these methods lie in the fact that they make determination of acceleration unnecessary. Methods of work and energy directly relate force, mass, velocity and displacement and enable us to explore motion between two points of space i.e. in a space interval whereas methods of impulse and momentum enable us to explore motion in a time interval. Moreover methods of impulse and momentum provides only way to analyze impulsive motion.

The work energy theorem and impulse momentum principle are developed from Newton's second law, and we have seen how to apply them to analyze motion of single particle i.e. translation motion of rigid body. Now we will further inquire into possibilities of applying these principles to a system of large number of particles or rigid bodies in translation motion.

System of Particles

By the term system of particles, we mean a well defined collection of several or large number of particles, which may or may not interact or be connected to each other.

As a schematic representation, consider a system of n particles of masses $m_1, m_2, \ldots m_i \ldots m_j \ldots$ and m_n respectively. They may be actual particles of rigid bodies in translation motion. Some of them may interact with each other and some of them may not. The particles, which interact with each other, apply forces on each other. The forces of interaction $\vec{f}_{ij}$ and $\vec{f}_{ji}$ between a pair of i^{th} and j^{th} particles are shown in the figure. Similar to these other particles may also interact with each other. These forces of mutual interaction between the particles of the system are *internal forces* of the system.



System of n interacting particles.

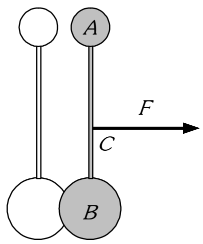
These internal forces always exist in pairs of forces of equal magnitudes and opposite directions. It is not necessary that all the particles interact with each other; some of them, which do not interact with each other, do not apply mutual forces on each other. Other than internal forces, external forces may also act on all or some of the particles. Here by the term *external force* we mean a force that is applied on any one of the particle included in the system by some other body out-side the system.

In practice we usually deal with extended bodies, which may be deformable or rigid. An extended body is also a system of infinitely large number of particles having infinitely small separations between them. When a body undergoes deformation, separations between its particles and their relative locations change. A rigid body is an extended body in which separations and relative locations of all of its particles remain unchanged under all circumstances.

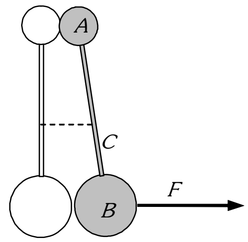
System of Particles and Mass Center

Until now we have deal with translation motion of rigid bodies, where a rigid body can be treated as a particle. When a rigid body undergoes rotation, all of its particles do not move in identical fashion, still we must treat it a system of particles in which all the particles are rigidly connected to each other. On the other hand we may have particles or bodies not connected rigidly to each other but may be interacting with each other through internal forces. Despite the complex motion of which a system of particles is capable, there is a single point, known as **center of mass** or **mass center** (CM), whose translation motion is characteristic of the system.

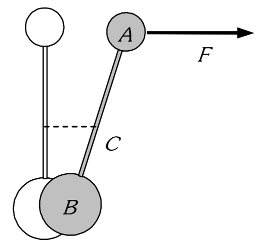
The existence of this special point can be demonstrated in the following examples dealing with a rigid body. Consider two disks A and B of unequal masses connected by a very light rigid rod. Place it on a very smooth table. Now pull it horizontally applying a force at different points. You will find a point nearer to the heavier disk, on which if the force is applied the whole assembly undergoes translation motion. Furthermore you cannot find any other point having this property. This point is the mass center of this system. We can assume that all the mass were concentrated at this point. In every rigid body we can find such a point. If you apply the force on any other point, the system moves forward and rotates but the mass center always translates in the direction of the force.



Force applied on the mass center

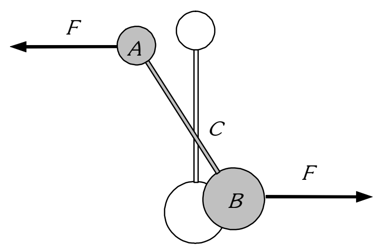


Force not applied on the mass center



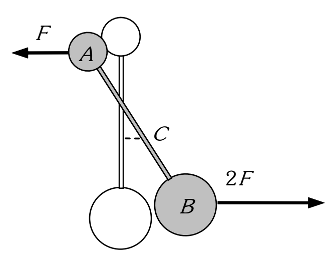
Force not applied on the mass center

In another experiment, if two forces of equal magnitudes are applied on the disks in opposite directions, the system will rotate, but the mass center C remains stationary as shown in the following figure.



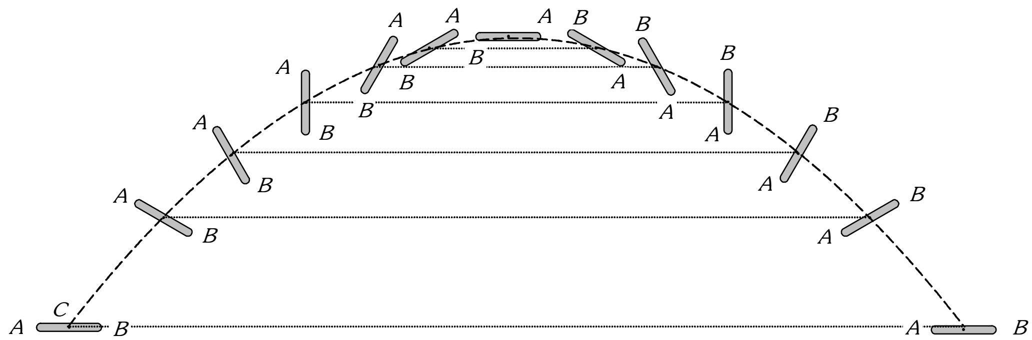
Body rotates but the mass center remains stationary under action of equal and opposite forces.

If the above experiment is repeated with both disks A and B of identical masses, the mass center will be the mid point. And if the experiment is repeated with a uniform rod, the mass center again is the mid point.



Body rotates and the mass center translates under action of unbalanced forces applied at different points.

As another example let us throw a uniform rod in air holding it from one of its ends so that it rotates also. Snapshots taken after regular intervals of time are shown in the figure. The rod rotates through 360 . As the rod moves all of its particles move in a complex manner except the mass center C, which follows a parabolic trajectory as if it were a particle of mass equal to that of the rod and force of gravity were acting on it.



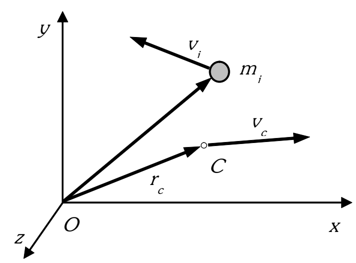
Thus mass center of a rigid body or system of particles is a point, whose translation motion under action of unbalanced forces is same as that of a particle of mass equal to that of the body or system under action the same unbalanced forces. And if different forces having a net resultant are applied at different particles, the system rotates but the mass center translates as if it were a particle of the mass same as that of the system and the net resultant were applied on it.

Concept of mass center provides us a way to look into motion of the system as a whole as superposition of translation of the mass center and motion of all the particles relative to the mass center. In case of rigid bodies all of its particles relative to the mass center can move only on circular paths because they cannot changes their separations.

The concept of mass center is used to represent gross translation of the system. Therefore total linear momentum of the whole system must be equal to the linear momentum of the system due to translation of its mass center.

Center of Mass of System of Discrete Particles

A system of several particles or several bodies having finite separations between them is known as system of discrete particles. Let at an instant particles of such a system $m_{1,}$ m_2, ...m_i, ...and m_n are moving with velocities $\vec{v}_1$, $\vec{v}_1$, $\vec{v}_2$, ... $\vec{v}_i$,......and $\vec{v}_n$ at locations $\vec{r}_1$, $\vec{r}_2$, ... $\vec{r}_i$,......and $\vec{r}_n$ respectively. For the sake of simplicity only i^{th} particle and the mass center C are shown in the figure. The mass center C located at $\vec{r}_c$ is moving with velocity $\vec{v}_c$ at this instant.



As the mass center represents gross translation motion of the whole system, the total linear i.e. sum of linear momenta of all the particles must be equal to linear momentum of the whole mass due to translation of the mass center.

$$m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots\ldots\ldots\ldots\ldots + m_i\vec{v}_i + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + m_n\vec{v}_n = M\vec{v}_c$$

We can write the following equation in terms of masses and position vectors as an analogue to the above equation. This equation on differentiating with respect to time yields the above equation therefore can be thought as solution of the above equation.

$$m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots\ldots\ldots\ldots\ldots + m_i\vec{r}_i + \ldots\ldots\ldots\ldots\ldots\ldots\ldots + m_n\vec{r}_n = M\vec{r}_c$$

If $M = \Sigma m_i$ denotes total mass of the system, the above two equations can be written in short as

$$\Sigma m_i\vec{v}_i = M\vec{v}_c \qquad (1)$$

$$\Sigma m_i\vec{r}_i = M\vec{r}_c \qquad (2)$$

The above equation suggests location of mass center of a system of discreet particles.

$$\vec{r}_c = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots\ldots\ldots\ldots + m_i\vec{r}_i + \ldots\ldots\ldots\ldots\ldots\ldots + m_n\vec{r}_n}{M} = \frac{\Sigma m_i\vec{r}_i}{M} \quad (3)$$

Cartesian coordinate (x_c, y_c, z_c) of the mass center are components of the position vector $\vec{r}_c$ of the mass center.

$$x_c = \frac{\Sigma m_i x_i}{M}\ ;\ y_c = \frac{\Sigma m_i y_i}{M}\ ;\ z_c = \frac{\Sigma m_i z_i}{M} \quad (4)$$

Example

Center of Mass of Two Particle System

(a) Find expression of position vector of mass center of a system of two particles of masses m_1 and m_2 located at position vectors $\vec{r}_1$ and $\vec{r}_2$.

(b) Express Cartesian coordinates of mass center, if particle m_1 at point (x_1, y_1) and particle m_2 at point (x_2, y_2).

(c) If you assume origin of your coordinate system at the mass center, what you conclude regarding location of the mass center relative to particles.

(d) Now find location of mass center of a system of two particles masses m_1 and m_2 separated by distance r.

Solution.

(a) Consider two particles of masses m_1 and m_2 located at position vectors $\vec{r}_1$ and $\vec{r}_2$. Let their mass center C at position vector $\vec{r}_c$.

From eq. , we have

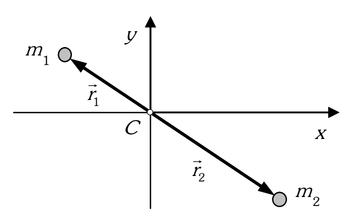
$$\vec{r}_c = \frac{\Sigma m_i\vec{r}_i}{M} \rightarrow \qquad \vec{r}_c = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

(b) From result obtained in part *(a)*, we have

$$x_c = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} \text{ and } y_c = \frac{m_1y_1 + m_2y_2}{m_1 + m_2}$$

(c) If we assume origin at the mass center vector $\vec{r}_c$ vanishes and we have

$$m_1\vec{r}_1 + m_2\vec{r}_2 = \vec{0}$$



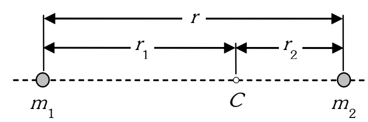
Since either of the masses m_1 and m_2 cannot be negative, to satisfy the above equation, vectors $\vec{r}_1$ and $\vec{r}_2$ must have opposite signs. It is geometrically possible only when mass center C lies between the two particles on the line joining them as shown in the figure.

If we substitute magnitudes r_1 and r_2 of vectors $\vec{r}_1$ and $\vec{r}_2$ in the above equation, we have $m_1r_1 = m_2r_2$, which suggest

$$\frac{r_1}{r_2} = \frac{m_2}{m_1}$$

Now we conclude that mass center of two particle system lies between the two particles on the line joining them and divide the distance between them in inverse ratio of masses of the particles.

(d) Consider two particles masses m_1 and m_2 at distance r from each other. There mass center C must lie in between them on the line joining them. Let distances of these particles from the mass center are r_1 and r_2.



Since mass center of two particle system lies between the two particles on the line joining them and divide the distance between them in inverse ratio of masses of the particles, we can write

$$r_1 = \frac{m_2 r}{m_1 + m_2} \text{ and } r_2 = \frac{m_1 r}{m_1 + m_2}$$

Example

Mass centre of several particles

Find position vectors of mass center of a system of three particle of masses 1 kg, 2 kg and 3 kg located at position vectors $\vec{r}_1 = \left(4\tilde{i} + 2\tilde{j} - 3\tilde{k}\right)$ m, $\vec{r}_2 = \left(\tilde{i} - 4\tilde{j} + 2\tilde{k}\right)$ m and $\vec{r}_3 = \left(2\tilde{i} - 2\tilde{j} + \tilde{k}\right)$ m respectively.

Solution.

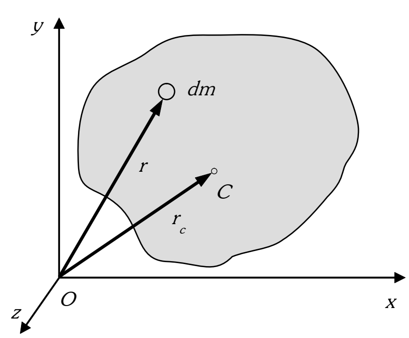
From eq. , we have

$$\vec{r}_c = \frac{\Sigma m_i \vec{r}_i}{M} \rightarrow \quad \vec{r}_c = \frac{1\left(4\tilde{i} + 2\tilde{j} - 3\tilde{k}\right) + 2\left(\tilde{i} - 4\tilde{j} + 2\tilde{k}\right) + 3\left(2\tilde{i} - 2\tilde{j} + \tilde{k}\right)}{1 + 2 + 3} = 2\tilde{i} - 2\tilde{j} + \frac{2}{3}\tilde{k}$$

Center of Mass of an Extended Body or Continuous Distribution of Mass

An extended body is collection of infinitely large number of particles so closely located that we neglect separation between them and assume the body as a continuous distribution of mass. A rigid body is an extended body in which relative locations of all the particles remain unchanged under all circumstances. Therefore a rigid body does not get deformed under any circumstances.

Let an extended body is shown as a continuous distribution of mass by the shaded object in the figure. Consider an infinitely small portion of mass dm of this body. It is called a mass element and is shown at position given by position vector $\vec{r}$. Total mass M of the body is $M = \int dm$. The mass center C is assumed at position given by position vector $\vec{r}_c$. Position vector of centre of mass of such a body is given by the following equation.



$$\vec{r}_c = \frac{\int \vec{r} dm}{M} \quad (5)$$

Cartesian coordinate (x_C, y_C, z_C) of the mass center are components of the position vector $\vec{r}_c$ of the mass center.

$$x_c = \frac{\int x dm}{M} ; \; y_c = \frac{\int y dm}{M} ; \; z_c = \frac{\int z dm}{M} \quad (6)$$

Example

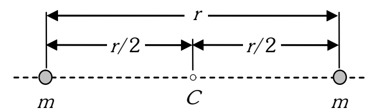
Mass centre of uniform symmetrical bodies.

Show that mass center of uniform and symmetric mass distributions lies on axis of symmetry.

Solution.

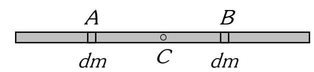
For simplicity first consider a system of two identical particles and then extend the idea obtained to a straight uniform rod, uniform symmetric plates and uniform symmetric solid objects.

Mass Center of a system of two identical particles



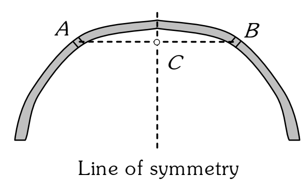
Mass center of a system of two identical particles lies at the midpoint between them on the lie joining them.

Mass Center of a system of a straight uniform rod



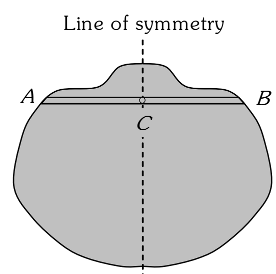
Consider two identical particles A and B at equal distances from the center C of the rod. Mass center of system these two particles is at C. The whole rod can be assumed to be made of large number of such systems each having its mass center at the mid point C of the rod. Therefore mass center of the whole rod must be at its mid point.

Mass Center of a system of a uniform symmetric curved rod



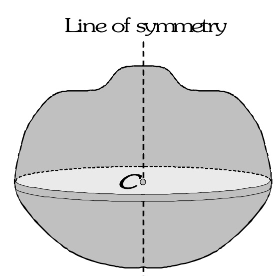
Consider two identical particles A and B located at equal distances from the line of symmetry. Mass center of system these two particles is at C. The whole rod can be assumed to be made of large number of such systems each having its mass center at the mid point C of the joining them. Therefore mass center of the whole rod must be on the axis of symmetry.

Mass Center of a uniform plate (lamina)



Consider a symmetric uniform plate. It can be assumed composed of several thin uniform parallel rods like rod AB shown in the figure. All of these rods have mass center on the line of symmetry, therefore the whole lamina has its mass center on the line of symmetry.

Mass Center of a uniform symmetric solid object



A uniform symmetric solid object occupies a volume that is made by rotating a symmetric area about its line of symmetry though 180°. Consider a uniform symmetric solid object shown in the figure. It can be assumed composed of several thin uniform parallel disks shown in the figure. All of these disks have mass center on the line of symmetry, therefore the whole solid object has its mass center on the line of symmetry.

Mass Center of uniform bodies

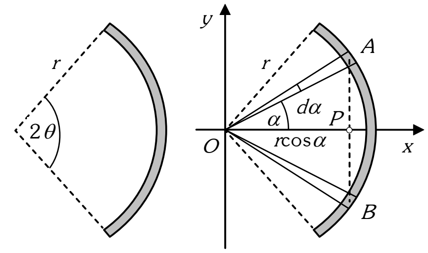
Following the similar reasoning, it can be shown that mass center of uniform bodies lies on their geometric centers.

Example

Mass Center of a system of a segment of a uniform circular rod (arc)

Find location of mass center of a thin uniform rod bent into shape of an arc.

Solution.



Consider a thin rod of uniform line mass density λ (mass per unit length) and radius r subtending angle 2θ on its center O.

The angle bisector OP is the line of symmetry, and mass center lies on it. Therefore if we assume the angle bisector as one of the coordinate axes say x-axis, y-coordinate of mass center becomes zero.

Let two very small segments A and B located symmetric to the line of symmetry (x-axis). Mass center of these two segments is on P at a distance $x = r\cos\alpha$ from center O. Total mass of these two elements is $dm = 2\lambda r d\alpha$. Now using eq. , we have

$$x_c = \frac{\int x dm}{M} \rightarrow \qquad x_c = \frac{\int_{-\theta}^{\theta}(r\cos\alpha)(2\lambda r d\alpha)}{\lambda r\theta} = \frac{r\sin\theta}{\theta}$$

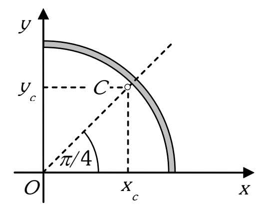
Mass center of a thin uniform arc shaped rod of radius r subtending angle 2θ at the center lies on its angle bisector at distance OC from the center.

$$OC = \frac{r\sin\theta}{\theta}$$

Example

Find coordinates of mass center of a quarter ring of radius r placed in the first quadrant of a Cartesian coordinate system, with centre at origin.

Solution.



Making use of the result obtained in the previous example, distance

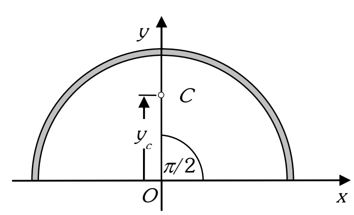
OC of the mass center form the center is $OC = \frac{r\sin(\pi/4)}{\pi/4} = \frac{2\sqrt{2}r}{\pi}$

Coordinates of the mass center (x_c, y_c) are $\left(\frac{2r}{\pi}, \frac{2r}{\pi}\right)$

Example

Find coordinates of mass center of a semicircular ring of radius r placed symmetric to the y-axis of a Cartesian coordinate system.

Solution.



The y-axis is the line of symmetry, therefore mass center of the ring lies on it making x-coordinate zero.

Distance OC of mass center from center is given by the result obtained in example 4. Making use of this result, we have

$$OC = \frac{r\sin\theta}{\theta} \rightarrow \qquad y_c = \frac{r\sin(\pi/2)}{\pi/2} = \frac{2r}{\pi}$$

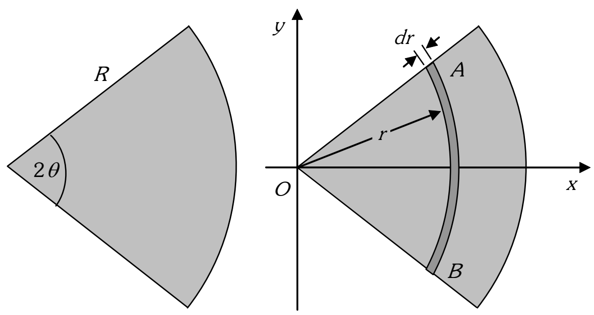
Example

Mass Center of a sector of a uniform circular plate

Find location of mass center of a sector of a thin uniform plate.

Solution.

Consider a sector of a thin uniform plate of surface mass density mass density σ (mass per unit area) and radius r subtending angle 2θ on its center.



Let a thin arc of radius r and width dr be an infinitely small part of the sector. Mass dm of the arc AB equals to product of mass per unit area and area of the arc.

$$dm = \sigma(2r\theta dr) = 2\sigma r\theta dr$$

Due to symmetry mass center of this arc must be on the angle bisector i.e. on x-axis at distance $x = \dfrac{r\sin\theta}{\theta}$.

Using above two information in eq. , we obtain the mass center of the sector.

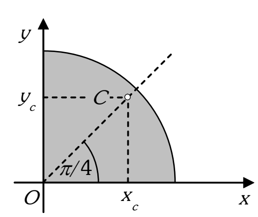
$$x_c = \frac{\int x dm}{M} \rightarrow \qquad x_c = \frac{\int_0^R \left(\frac{r\sin\theta}{\theta}\right)(\sigma r\theta dr)}{\sigma(\text{Area of thre sector})} = \frac{\int_0^R \left(\frac{r\sin\theta}{\theta}\right)(2\sigma r\theta dr)}{\sigma r\theta^2} = \frac{2r\sin\theta}{3\theta}$$

Example

Find coordinates of mass center of a quarter sector of a uniform disk of radius r placed in the first quadrant of a Cartesian coordinate system with centre at origin.

Solution.

Making use of the result obtained in the previous example, distance OC of the mass center form the center is



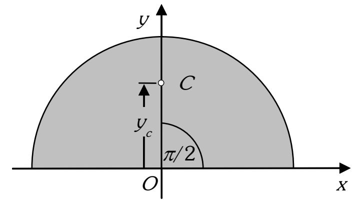
$$OC = \frac{2r\sin(\pi/4)}{3\pi/4} = \frac{4\sqrt{2}r}{3\pi}$$

Coordinates of the mass center (x_c, y_c) are $\left(\dfrac{4r}{3\pi}, \dfrac{4r}{3\pi}\right)$

Example

Find coordinates of mass center of a uniform semicircular plate of radius r placed symmetric to the y-axis of a Cartesian coordinate system, with centre at origin.

Solution.



The y-axis is the line of symmetry, therefore mass center of the plate lies on it making x-coordinate zero.

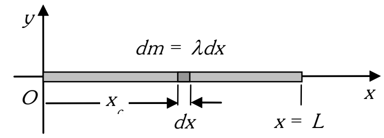
Distance OC of mass center from center is given by the result obtained in example 7. Making use of this result, we have

$$OC = \frac{2r\sin\theta}{3\theta} \rightarrow \quad y_c = \frac{2r\sin(\pi/2)}{3\pi/2} = \frac{4r}{3\pi}$$

Example

Find coordinates of mass center of a non-uniform rod of length L whose linear mass density λ varies as $\lambda=a+bx$, where x is the distance from the lighter end.

Solution.



Assume the rod lies along the x-axis with its lighter end on the origin to make mass distribution equation consistent with coordinate system.

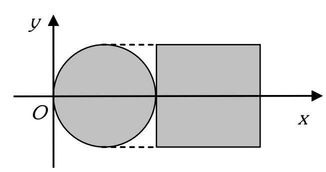
Making use of eq. , we have $x_c = \dfrac{\int x\,dm}{M} \rightarrow x_c = \dfrac{\int_0^L x\lambda dx}{\int_0^L \lambda dx} = \dfrac{\int_0^L x(ax+b)dx}{\int_0^L (ax+b)dx} = \dfrac{(2aL+3b)L}{3(aL+2b)}$

Example

Mass Center of composite bodies

A composite body is made of joining two or more bodies. Find mass center of the following composite body made by joining a uniform disk of radius r and a uniform square plate of the same mass per unit area.

Solution.



To find mass center the component bodies are assumed particle of masses equal to corresponding bodies located on their respective mass centers. Then we use equation to find coordinates of the mass center of the composite body.

To find mass center of the composite body, we first have to calculate masses of the bodies, because their mass distribution is given.

If we denote surface mass density (mass per unit area) by σ, masses of the bodies are

Mass of the disk $\quad m_d$ = Mass per unit area × Area $= \sigma(\pi r^2) = \sigma\pi r^2$

Mass of the square plate $\quad m_p$ = Mass per unit area × Area $= \sigma(r^2) = \sigma r^2$

Location of mass center of the disk $\quad x_d$ = Center of the disk $= r$ and $y_d = 0$

Location of mass center of the square plate $\quad x_p$ = Center of the surface plate $= 3r$ and $y_d = 0$

Using eq. , we obtain coordinates (x_c, y_c) of the composite body.

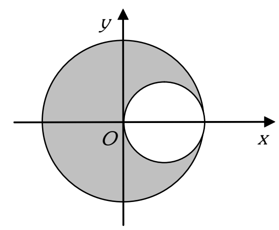
$$x_c = \frac{m_d x_d + m_s x_s}{m_d + m_s} = \frac{r(\pi+3)}{(\pi+1)} \quad \text{and} \quad y_c = \frac{m_d x_d + m_s x_s}{m_d + m_s} = 0$$

Coordinates of the mass center are $\left(\dfrac{r(\pi+3)}{(\pi+1)},\ 0\right)$

Example

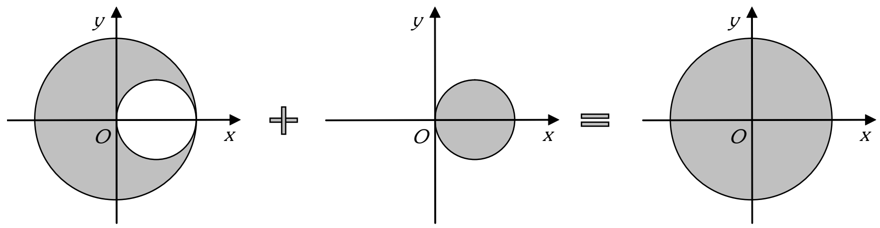
Mass Center of truncated bodies

A truncated body is made by removing a portion of a body. Find mass center of the following truncated disk made by removing disk of radius equal to half of the original disk as shown in the figure. Radius of the original uniform disk is r.



Solution.

To find mass center of truncated bodies we can make use of superposition principle that is, if we add the removed portion in the same place we obtain the original body. The idea is illustrated in the following figure.



The removed portion is added to the truncated body keeping their location unchanged relative to the coordinate frame.

Denoting masses of the truncated body, removed portion and original body by m_{tb}, m_{rp} and m_{ob} and location of their mass centers by x_{tb}, x_{rp} and x_{ob}, we can write $m_{tb}x_{tb} + m_{rb}x_{rp} = m_{ob}x_{ob}$

From the above equation we obtain position co-ordinate x_{tb} of the mass center of the truncated body.

$$x_{tb} = \frac{m_{ob}x_{ob} - m_{rb}x_{rp}}{m_{tb}} \qquad (1)$$

Denoting mass per unit area by σ, we can express the masses m_{tb}, m_{rp} and m_{ob}.

Mass of truncated body $\qquad m_{tb} = \sigma\left\{\pi\left(r^2 - \frac{r^2}{4}\right)\right\} = \frac{3\sigma\pi r^2}{4}$

Mass of the removed portion $\qquad m_{rp} = \frac{\sigma\pi r^2}{4}$

Mass of the original body $\qquad m_{ob} = \sigma\pi r^2$

Mass center of the truncated body $\qquad x_{tb}$

Mass center of the removed portion $\qquad x_{rp} = \frac{r}{2}$

Mass center of the original body $\qquad x_{ob} = 0$

Substituting the above values in equation (1), we obtain the mass the center of the truncated body.

$$x_{tb} = \frac{m_{ob}x_{ob} - m_{rb}x_{rp}}{m_{tb}} = \frac{\left(\sigma\pi r^2\right)\times 0 - \left(\frac{\sigma\pi r^2}{4}\right)\left(\frac{r}{2}\right)}{\frac{3\sigma\pi r^2}{4}} = -\frac{r}{6}$$

Mass center of the truncated body is at point $\left(-\frac{r}{6},\ 0\right)$

Center of Mass Frame of Reference or Centroidal Frame

Center of mass frame of reference or centroidal frame is reference frame assume attached with the mass center of the system at its origin. It moves together with the mass center.

It is a special frame and presents simple interpretations and solutions to several phenomena. Let us first discuss some of its fundamental properties.

In centroidal frame center of mass is assumed at the origin, therefore position vector, velocity and acceleration of the mass center in centroidal frame all become zero.

- Sum of mass moments in centroidal frame vanishes.

  Mass moment of a particle is product of mass of the particle and its position vector.

$$\Sigma m_i \vec{r}_{i/c} = \vec{0} \text{ or } m_1\vec{r}_{1/c} + m_2\vec{r}_{2/c} + \ldots\ldots\ldots\ldots + m_i\vec{r}_{i/c} + \ldots\ldots\ldots\ldots\ldots\ldots + m_n\vec{r}_{n/c} = \vec{0} \qquad (7)$$

- Total linear momentum of the system in centroidal frame vanishes..

$$\Sigma m_i \vec{v}_{i/c} = \vec{0} \text{ or } m_1\vec{v}_{1/c} + m_2\vec{v}_{2/c} + \ldots\ldots\ldots\ldots + m_i\vec{v}_{i/c} + \ldots\ldots\ldots\ldots\ldots\ldots + m_n\vec{v}_{n/c} = \vec{0} \qquad (8)$$

Example

Motion of Mass Center in One Dimension

A jeep of mass 2400 kg is moving along a straight stretch of road at 80 km/h. It is followed by a car of mass 1600 kg moving at 60 km/h.

(a) How fast is the center of mass of the two cars moving?

(b) Find velocities of both the vehicles in centroidal frame.

Solution.

(a) Velocity of the mass center $\vec{v}_c = \dfrac{m_{jeep}\vec{v}_{jeep} + m_{car}\vec{v}_{car}}{m_{jeep} + m_{car}}$

Assuming direction of motion in the positive x-direction, we have

$$\vec{v}_c = \frac{m_{jeep}\vec{v}_{jeep} + m_{car}\vec{v}_{car}}{m_{jeep} + m_{car}} \rightarrow \quad \vec{v}_c = \frac{2400 \times 80 + 1600 \times 60}{2400 + 1600} = 72 \text{ km/h}$$

(b) Velocity of the jeep in centroidal frame $v_{jeep/c} = 80 - 72 = 8$ km/h in positive x-direction.

Velocity of the car in centroidal frame $v_{car/c} = 60 - 72 = -12$ km/h

12 km/h negative x-direction direction.

Example

Motion of Mass Center in Vector Form

A 2.0 kg particle has a velocity of $\vec{v}_1 = \left(2.0\tilde{i} - 3.0\tilde{j}\right)$ m/s, and a 3.0 kg particle has a velocity $\vec{v}_2 = \left(1.0\tilde{i} + 6.0\tilde{j}\right)$ m/s.

(a) How fast is the center of mass of the particle system moving?

(b) Find velocities of both the particles in centroidal frame.

Solution.

(a) Velocity of the mass center $\vec{v}_c = \dfrac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$

$$\vec{v}_c = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2} \rightarrow \qquad \vec{v}_c = \frac{2\left(2.0\tilde{i} - 3.0\tilde{j}\right) + 3\left(1.0\tilde{i} + 6.0\tilde{j}\right)}{2+3} = \left(1.4\tilde{i} + 2.4\tilde{j}\right) \text{ m/s}$$

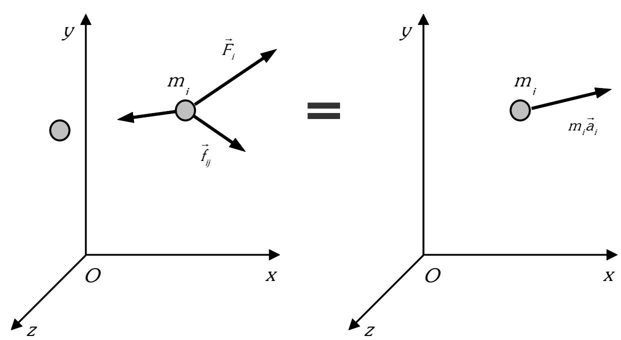
(b) Velocity of the first particle in centroidal frame

$$\vec{v}_{1/c} = \vec{v}_1 - \vec{v}_c \rightarrow \qquad \vec{v}_{1/c} = \left(2.0\tilde{i} - 3.0\tilde{j}\right) - \left(1.4\tilde{i} + 2.4\tilde{j}\right) = 0.6\left(\tilde{i} + \tilde{j}\right) \text{ m/s}$$

Velocity of the second particle in centroidal frame

$$\vec{v}_{2/c} = \vec{v}_1 - \vec{v}_c \rightarrow \qquad \vec{v}_{2/c} = \left(1.0\tilde{i} + 6.0\tilde{j}\right) - \left(1.4\tilde{i} + 2.4\tilde{j}\right) = -\left(0.4\tilde{i} + 3.6\tilde{j}\right) \text{ m/s}$$

Application of Newton's Laws of Motion to a System of Particles



In order to write equation of motion for a system of particles, we begin by applying Newton's second law to an individual particle.

Consider i^{th} particle of mass m_i. Internal force applied on it by j^{th} particle is shown by $\vec{f}_{ij}$. Other particles of the system may also apply internal forces on it. One of them is shown in the figure by an unlabeled vector. In addition to these internal forces, external forces may also be applied on it by bodies out side the system. Resultant of all these external forces is shown by vector $\vec{F}_i$. If under the action of these forces this particles has acceleration $\vec{a}_i$ relative to an inertial frame *Oxyz*, its free body diagram and kinetic diagram can be represented by the following figure and Newton's second law can be written by the following equation.

$$\vec{F}_i + \sum \vec{f}_{ij} = m_i\vec{a}_i$$

In similar fashion, we can write Newton's second law for all the particles of the system. These equations are

For 1^{st} particle $\qquad \vec{F}_1 + \sum \vec{f}_{1j} = m_1\vec{a}_1$

For 2^{nd} particle $\qquad \vec{F}_2 + \sum \vec{f}_{2j} = m_2\vec{a}_2$

………………… $\qquad$ …………………

For i^{th} particle $\qquad \vec{F}_i + \sum \vec{f}_{ij} = m_i\vec{a}_i$

………………… $\qquad$ …………………

For n^{th} particle $\qquad \vec{F}_n + \sum \vec{f}_{nj} = m_n\vec{a}_n$

Every internal force $\vec{f}_{ij}$ on particle m_i due to particle m_j and $\vec{f}_{ji}$ on the particle m_j due to particle m_i constituting Newton's third law pair must be equal in magnitude and opposite in direction, therefore the sum all these internal forces for all the particles must be zero. Keeping this fact in mind and denoting the mass of the whole system by M and acceleration of the mass center C by $\vec{a}_C$ relative to the inertial frame, Newton's second law representing translation motion of the system of particles particle can be represented by the following equation.

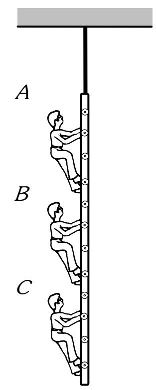
$$\sum \vec{F}_i = \sum \left(m_i \vec{a}_i\right) = M\vec{a}_C \qquad (9)$$

$$\sum \vec{F}_i = M\vec{a}_C = \frac{d\vec{p}_c}{dt} \qquad (10)$$

Example

Newton' Laws of Motion and System of Particles

A ladder of mass 20 kg is hanging from ceiling as shown in figure. Three men A, B and C of masses 40 kg, 60 kg, and 50 kg are climbing the ladder. Man A is climbing with upward retardation 2 m/s^2, B is climbing up with a constant speed of 0.5 m/s and C is climbing with upward acceleration of 1 m/s^2. Find the tension in the string supporting the ladder.



Solution.

External forces acting on the system are weights of the men, weight of the ladder and tension supporting the ladder. Denoting masses of men A, B, C and ladder by m_A, m_B, m_C and m_L, acceleration due to gravity by g , tension in the string by T and accelerations of the men A, B, C and ladder by a_A, a_B, a_C and a_L respectively, we can write the following equation according to equation .

$$\Sigma \vec{F}_i = \Sigma\left(m_i \vec{a}_i\right) \rightarrow \qquad T - m_A g - m_B g - m_C g - m_L g = m_A a_A + m_B a_A + m_C a_A + m_L a_A$$

Substituting given values of masses $m_A = 40$ kg, $m_B = 60$ kg, $m_C = 50$ kg, $m_L = 20$ kg,

given values of accelerations $g = 10$ m/s^2, $a_A = -2$ m/s^2, $a_B = 0$ m/s^2, $a_C = 1$ m/s^2, and $a_L = 0$ m/s^2,

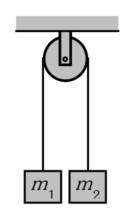
we obtain $\qquad T - 400 - 600 - 500 - 200 = -80 + 0 + 50 + 0$

$T = 1670$ N

Example

Simple Atwood Machine as System of Particles

The system shown in the figure is known as simple Atwood machine. Initially the masses are held at rest and then let free. Assuming mass m_2 more than the mass m_1, find acceleration of mass center and tension in the string supporting the pulley.



Solution.

We know that accelerations a_1 and a_2 are given by the following equations.

$$a_2 = \frac{m_2 - m_1}{m_2 + m_1} g \downarrow \quad \text{and} \quad a_1 = \frac{m_2 - m_1}{m_2 + m_1} g \uparrow$$

Making use of eq. , we can find acceleration a_C of the mass center. We denote upward direction positive and downward direction negative signs respectively.

$$M\vec{a}_C = \Sigma\left(m_i \vec{a}_i\right) \rightarrow \qquad \left(m_1 + m_2\right) a_c = m_1 a_1 - m_2 a_2$$

Substituting values of accelerations a_1 and a_2, we obtain

$$a_C = \frac{2m_1m_2 - \left(m_1^2 + m_2^2\right)}{\left(m_1 + m_2\right)^2} g$$

To find tension T in the string supporting the pulley, we again use eq. (9)

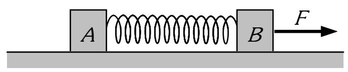
$\Sigma \vec{F}_i = M\vec{a}_C \rightarrow$ $\quad T - m_1g - m_2g = \left(m_1 + m_2\right)a_c$

Substituting expression obtained for a_C, we have

$$T = \frac{4m_1m_2}{m_1 + m_2} g$$

Example

Two blocks each of mass m, connected by an un-stretched spring are kept at rest on a frictionless horizontal surface. A constant force F is applied on one of the blocks pulling it away from the other as shown in figure.



(a) Find acceleration of the mass center.

(b) Find the displacement of the centre of mass as function of time t.

(c) If the extension of the spring is x_o at an instant t, find the displacements of the two blocks relative to the ground at this instant.

Solution.

(a) Forces in vertical direction on the system are weights of the blocks and normal reaction from the ground. They balance themselves and have no net resultant. The only external force on the system is the applied force F in the horizontal direction towards the right.

$\Sigma \vec{F}_i = M\vec{a}_C \rightarrow$ $\quad F = (m + m)a_c$

$$a_c = \frac{F}{2m} \text{ towards right}$$

(b) The mass center moves with constant acceleration, therefore it displacement in time t is given by equation of constant acceleration motion.

$x = ut + \frac{1}{2}at^2 \rightarrow$ $\quad x_c = \frac{Ft^2}{4m}$

(c) Positions x_A and x_B of particles A and B forming a system and position x_C mass center are obtained by following eq.

$$M\vec{r}_c = \Sigma m_i\vec{r}_i$$

Substituting values wee obtain $2mx_c = mx_A + mx_B$ $\quad x_c = \frac{x_A + x_B}{2}$

Now using result obtained in part *(b)*, we have $\quad x_A + x_B = \frac{Ft^2}{2m}$

Extension in the spring at this instant is $x_o = x_B - x_A$

From the above two equations, we have $\quad x_A = \frac{1}{2}\left(\frac{Ft^2}{2m} - x_0\right)$ and $x_B = \frac{1}{2}\left(\frac{Ft^2}{2m} + x_0\right)$

Application of Methods of Impulse and Momentum to a System of Particles

In a phenomenon, when a system changes its configuration, some or all of its particles change their respective locations and momenta. Sum of linear momenta of all the particles equals to the linear momentum due to translation of mass center. Principle of impulse and moment suggests net impulse of all the external forces equals to change in momentum of mass center.

$$\sum \int \vec{F}_i dt = \vec{p}_{cf} - \vec{p}_{ci} \qquad (11)$$

Conservation of Linear momentum

The above event suggests that total linear momentum of a system of particle remains conserved in a time interval in which impulse of external forces is zero.

Total momentum of a system of particles cannot change under the action of internal forces and if net impulse of the external forces in a time interval is zero, the total momentum of the system in that time interval will remain conserved.

$$\sum \vec{p}_{initial} = \sum \vec{p}_{final} \text{ or } \vec{p}_{ci} = \vec{p}_{cf} \qquad (12)$$

The above statement is known as the *principle of conservation of momentum.*

Since force, impulse and momentum are vectors, component of momentum of a system in a particular direction is conserved, if net impulse of all external forces in that direction vanishes.

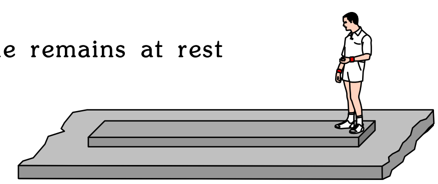
During an event the net impulse of external forces in a direction is zero in the following cases.

- When no external force acts in a particular direction on any of the particles or bodies.
- When resultant of all the external forces acting in a particular direction on all the particles or bodies is zero.
- In impulsive motion, where time interval is negligibly small, the direction in which no impulsive forces act.

Example

No external force: Stationary mass relative to an inertial frame remains at rest

A man of mass *m* is standing at on end of a plank of mass *M*. The length of the plank is *L* and it rests on a frictionless horizontal ground. The man walks to the other end of the plank. Find displacement of the plank and man relative to the ground.



Solution.

Denoting *x*-coordinates of the man, mass center of plank and mass center of the man-plank system by x_m, x_p and x_c, we can write the following equation.

$$\left(\Sigma m_i\right)\vec{r}_c = \Sigma m_i \vec{r}_i \rightarrow \qquad (m+M)\vec{x}_c = m\vec{x}_m + M\vec{x}_p$$

Net force on the system relative to the ground is zero. Therefore mass center of the system which is at rest before the man starts walking, remains at rest $\left(\Delta\vec{x}_c = 0\right)$ after while the man walks on the plank.

$$\left(\Delta\vec{x}_c = 0\right) \rightarrow \qquad m\Delta\vec{x}_m + M\Delta\vec{x}_p = 0 \qquad (1)$$

The man walks displacement $\left(\Delta\vec{x}_{m/p} = -L\tilde{i}\right)$ relative to the plank. Denoting displacements of the man and the plank relative to the ground by $\Delta\vec{x}_m$ and $\Delta\vec{x}_p$, we can write

$$\Delta\vec{x}_{m/p} = \Delta\vec{x}_m - \Delta\vec{x}_p \rightarrow \quad \Delta\vec{x}_m - \Delta\vec{x}_p = -L\tilde{i} \qquad (2)$$

From the above equations (1) and (2), we have

$$\Delta\vec{x}_m = -\frac{ML\hat{i}}{m+M}$$

The man moves a distance $\frac{ML}{m+M}$ towards left relative to the ground.

$$\Delta\vec{x}_p = \frac{mL\hat{i}}{m+M}$$

The plank moves a distance $\frac{mL}{m+M}$ towards right relative to the ground.

Example

No external force: Mass center moving relative to an inertial frame moves with constant velocity

Two particles of masses 2 kg and 3 kg are moving under their mutual interaction in free space. At an instant they were observed at points (–2 m, 1 m, 4 m) and (2 m, –3 m, 6 m) with velocities $\left(3\hat{i}-2\hat{j}+\hat{k}\right)$ m/s and $\left(-\hat{i}+\hat{j}-2\hat{k}\right)$ m/s respectively. If after 10 sec, the first particle passes the point (6 m, 8 m, –6 m), find coordinate of the point where the second particle passes at this instant?

Solution.

System of these two particles is in free, therefore no external forces act on them. There total linear momentum remains conserved and their mass center moves with constant velocity relative to an inertial frame.

Velocity of the mass center

$$v_c = \frac{\Sigma m_i\vec{v}_i}{\Sigma m_i} = \frac{2\left(3\hat{i}-2\hat{j}+\hat{k}\right)+3\left(-\hat{i}+\hat{j}-2\hat{k}\right)}{2+3} = \frac{3\hat{i}-\hat{j}-4\hat{k}}{5}\ \text{m/s}$$

Location $\vec{r}_{co}$ of the mass center at the instant t = 0 s

$$\vec{r}_c = \frac{\Sigma m_i\vec{r}_i}{\Sigma m_i} \rightarrow \qquad \vec{r}_{co} = \frac{2\left(-2\hat{i}+\hat{j}+4\hat{k}\right)+3\left(2\hat{i}-3\hat{j}+6\hat{k}\right)}{2+3} = \frac{2\hat{i}-7\hat{j}+26\hat{k}}{5}$$

New location $\vec{r}_c$ of the mass center at the instant t = 10 s

$$\vec{r}_c = \vec{r}_{co} + \vec{v}_c t \rightarrow \qquad \vec{r}_c = \frac{2\hat{i}-7\hat{j}+26\hat{k}}{5} + \frac{3\hat{i}-\hat{j}-4\hat{k}}{5}\times 10 = \frac{32\hat{i}-17\hat{j}-14\hat{k}}{5}$$

New location (x, y, z) of the second particle.

$$\vec{r}_c = \frac{\Sigma m_i\vec{r}_i}{\Sigma m_i} \rightarrow \qquad \frac{32\hat{i}-17\hat{j}-14\hat{k}}{5} = \frac{2\left(6\hat{i}+8\hat{j}-6\hat{k}\right)+3\left(x\hat{i}+y\hat{j}+z\hat{k}\right)}{2+3}$$

Solving the above equation, we obtain the coordinates of the second particle (20/3, –11, –2/3)

Application of Methods of Work and Energy to a System of Particles

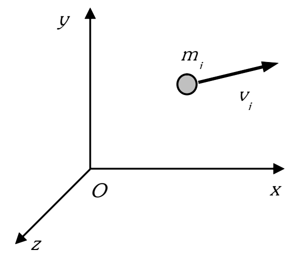
In a system of particles, all the particles occupy different locations at every instant of time and may change their locations with time. At an instant of time set of locations of all the particles of a system is known as *configuration* of the system. We say something has happened with the system only when some or all of its particles change their locations. It means that in every event or phenomena the system changes its configuration.

Methods of work and energy equips us to analyze what happens when a particle moves form one point of space to other. Now we will apply these methods to analyze a phenomenon in which a system of particle changes its configuration.

Kinetic Energy of a System of Particle

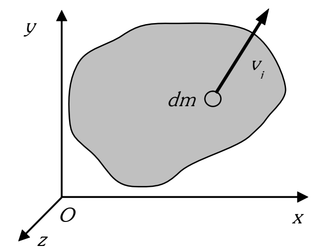
Kinetic energy of a system of particles is defined as sum of kinetic energies of all the particles of the system.

If at an instant particles of masses m_1, m_2, ...m_i....m_j.......... and m_n are observed moving with velocities $\vec{v}_1$, $\vec{v}_2$, ... $\vec{v}_i$, ...$\vec{v}_n$ respectively relative to a reference frame, the kinetic energy of the whole system relative to the reference frame is given by the following equation.



$$K = \frac{1}{2}\sum m_i v_i^2 \qquad (13)$$

If the system consists of continuous distribution of mass, instead of discrete particles, expression of kinetic energy becomes

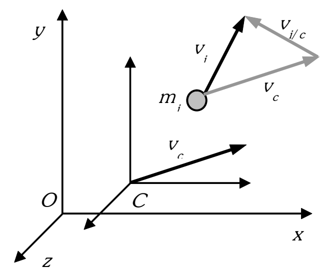


$$K = \frac{1}{2}\int v^2 dm \qquad (14)$$

Kinetic Energy of a System of Particle using Centroidal Frame

Centroidal frame of reference or center of mass frame is reference frame attached with the mass center of the system.

Let velocity of i^{th} particle of mass m_i is moving with velocity $\vec{v}_i$ relative to frame *Oxyz.* Mass center *C* and hence the centroidal frame *Cxyz* is moving with velocity $\vec{v}_c$. Therefore velocity of i^{th} particle relative to the centroidal frame is $\vec{v}_{i/c}$.



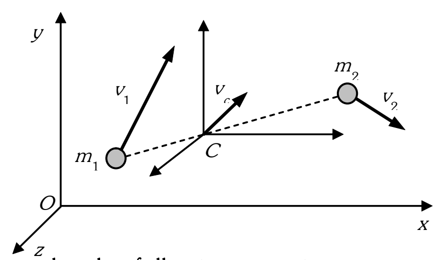
Kinetic energy of the whole system is given by the following equation.

$$K = \frac{1}{2}\sum m_i v_i^2 = \frac{1}{2}\sum m_i v_c^2 + \frac{1}{2}\sum m_i v_{i/c}^2 \qquad (15)$$

Here the first term on the right hand side is kinetic energy due to translation of the mass center and the second term is kinetic energy of the system relative to the centroidal frame.

Kinetic Energy of a Two Particle System using Centroidal Frame

A two particle system consists of only two particles. Let a two particle system consists of particles of masses m_1 and m_2 moving with velocities $\vec{v}_1$ and $\vec{v}_2$ relative to a frame *Oxyz*. Their mass center *C* lies on the line joining them and divides separation between them in reciprocal ratio of masses m_1 and m_2. The mass center and hence the centroidal frame is moving with velocity $\vec{v}_c$.



Kinetic energy of this two particle system relative to a frame *Oxyz* is given by the following equation.

$$K = \frac{1}{2}(m_1 + m_2)v_c^2 + \frac{1}{2}\mu v_{rel}^2 \qquad (16)$$

The first term on the right hand side is kinetic energy due to translation of the mass center and the second term is kinetic energy of the system relative to the centroidal frame.

Here symbol μ is known as reduced mass of the two particle system and symbol v_{rel} is magnitude of velocity of either of the particles relative to the other.

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad \text{and} \quad v_{rel} = |\vec{v}_1 - \vec{v}_2| = |\vec{v}_2 - \vec{v}_1|$$

Work Energy Theorem for a System of Particles

The work energy theorem can be applied to each particle of the system. For i^{th} particle of the system, we can write

$$K_{i,i} + W_{i,i\to f} = K_{i,f}$$

Here $W_{i,i'!f}$ is total work done by all the internal forces $\vec{f}_{ij}$ and resultant external force $\vec{F}_i$ on the i^{th} particle, when the system goes from one configuration to other.

Adding kinetic energies of all particles, we can write kinetic energies K_i and K_f of the whole system in the initial as well as the final configuration. Adding work done $W_{i,i\to f}$ by internal as well as external forces on every particle we find total work done $W_{i\to f}$ by all the internal as well as external forces on the system. Now we can write work energy theorem.

$$K_i + W_{i\to f} = K_f \qquad (17)$$

While applying the above equation to a system, care must be taken in calculating $W_{i\to f}$. In spite of the fact that the internal forces $\vec{f}_{ij}$ and $\vec{f}_{ji}$ being equal in magnitude and opposite in direction, the work done by them on the i^{th} and the j^{th} particles will not, in general, cancel out, since i^{th} and the j^{th} particles may undergo different amount of displacements.

The above description at first presents calculating of $W_{i\to f}$ as a cumbersome task. However for systems, which we usually encounter are not as complex as a general system of large number of particles may be. Systems which we usually face to analyze have limited number of particles or bodies interacting. For these systems we can simplify the task by calculating work of conservative internal forces as decrease in potential energy of the system. Total work of internal forces other than internal conservative forces vanishes, if these forces are due connecting inextensible links or links of constant length. These forces include string tension and normal reaction at direct contacts between the bodies included I the system. Work of internal forces of the kind other than these and work of external forces, can be calculated by definition of work.

Conservation of Mechanical Energy

If total work of internal forces other than conservative is zero and no external forces act on a system, total mechanical energy remains conserved.

$$K_i + U_i = K_f + U_f \qquad (18)$$

Since external forces are capable of changing mechanical energy of the system, under their presence total mechanical energy changes by amount equal to work $W_{ext,\ i'!f}$ done by all the external forces.

$$W_{ext,\ i\to f} = E_f - E_i = \left(K_f + U_f\right) - \left(K_i + U_i\right) \qquad (19)$$

Example

Total work of pseudo forces in centroidal frame.

Show that total work done in centroidal frame on all the particles of a system by pseudo forces due to acceleration of mass center is zero.

Solution.

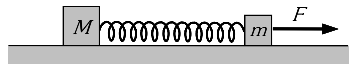
Let acceleration of mass centre relative to an inertial frame is $\vec{a}_c$. Pseudo force on i^{th} particle of mass m_i in centroidal frame is $\left(-m_i\vec{a}_c\right)$. Let displacement of i^{th} particle in a time interval is $\Delta\vec{r}_i$ relative to the centroidal frame.

Total work of pseudo forces on all the particles in centroidal frame can now be expressed by the expression

$$\Sigma\left(-m_i\vec{a}_c\right)\cdot\Delta\vec{r}_i = -\vec{a}_c\cdot\Sigma\left(m_i\Delta\vec{r}_i\right) = -\vec{a}_c\cdot\vec{0} = 0$$

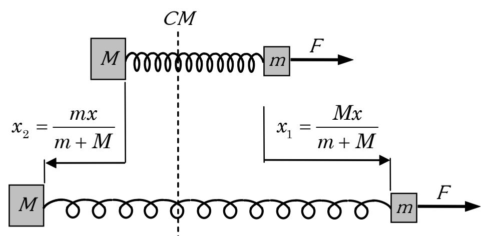
Example

Two blocks of masses m and M connected by a spring are placed on frictionless horizontal ground. When the spring is relaxed, a constant force F is applied as shown. Find maximum extension of the spring during subsequent motion.



Solution.

If we use ground as inertial frame as we usually do, solution of the problem becomes quite involved. Therefore, we prefer to use the centroidal frame, in which mass center remains at rest.



In the adjacent figure is shown horizontal position of mass center (CM) by dashed line. It remains unchanged in centroidal frame.

Mass center of two particle system divides separation between them in reciprocal ratio of the masses; therefore displacements x_1 and x_2 of the blocks must also be in reciprocal ratio of their masses. The extension x is sum of displacements x_1 and x_2 of the blocks as shown in the figure.

When extension of the spring achieves its maximum value, both the block must stop receding away from the mass center, therefore, velocities of both the blocks in centroidal frame must be zero.

During the process when spring is being extended, total work done by pseudo forces in centroidal frame become zero, negative work done by spring forces becomes equal to increase in potential energy and work done by the applied force evidently becomes Fx_1.

Using above fact in applying work energy theorem on the system relative to the centroidal frame, we obtain

$$K_i + W_{i\to f} = K_f \to \qquad 0 + W_{i\to f,springforce} + W_{i\to f,F} = 0$$

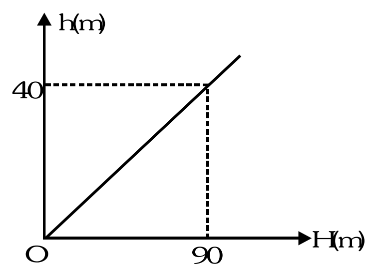
$$-\left(0 - \tfrac{1}{2}kx^2\right) + \frac{FMx}{m+m} = 0$$

$$x = \frac{2FM}{k(m+m)}$$

SOME WORKED OUT EXAMPLES

Example#1

A ball of mass 2 kg dropped from a height H above a horizontal surface rebounds to a height h after one bounce. The graph that relates H to h is shown in figure. If the ball was dropped from an initial height of 81 m and made ten bounces, the kinetic energy of the ball immediately after the second impact with the surface was



(A) 320 J (B) 480 J (C) 640 J (D) Can't be determined

Solution **Ans. (A)**

From graph $e = \sqrt{\frac{h}{H}} = \sqrt{\frac{40}{90}} = \frac{2}{3}$

Kinetic energy of the ball just after second bounce

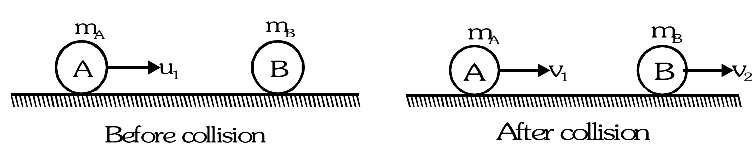
$$= \frac{1}{2}m\left(e^2u\right)^2 = \frac{1}{2}me^4u^2 = \left(e^4\right)\left(mgH\right) = \left(\frac{2}{3}\right)^4(2)(10)(81) = 320J$$

Example#2

Consider an one dimensional elastic collision between a given incoming body A and body B, initially at rest. The mass of B in comparison to the mass of A in order that B should recoil with greatest kinetic energy is

(A) $m_B >> m_A$ (B) $m_B << m_A$ (C) $m_B = m_A$ (D) can't say anything

Solution **Ans. (C)**



Velocity of block B after collision $v_2 = \frac{2m_Au_1}{m_A + m_B}$

KE of block B = $\frac{1}{2}m_Bv_2^2 = \frac{1}{2}m_B\left[\frac{4m_A^2u_1^2}{\left(m_A+m_B\right)^2}\right] = \frac{2m_A^2m_B}{\left(m_A+m_B\right)^2}u_1^2$

which is maximum if $m_A = m_B$

Example#3

An object is moving through air at a speed v. If the area of the object normal to the direction of velocity is A and assuming elastic collision with the air molecules, then the resistive force on the object is proportional to– (assume that molecules striking the object were initially at rest)

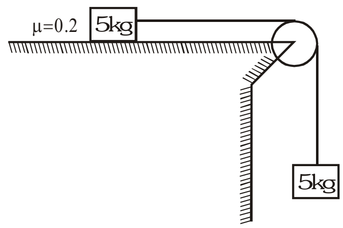
(A) 2Av (B) $2Av^2$ (C) $2Av^{1/2}$ (D) Can't be determined

Solution **Ans. (B)**

Velocity of air molecule after collision = 2v . The number of air– molecules accelerated to a velocity 2v in time Δt is proportional to $Av\Delta t$. Therefore $F = \dfrac{\Delta p}{\Delta t} \propto (Av\Delta t)\left(\dfrac{2v}{\Delta t}\right) \Rightarrow F \propto 2Av^2$

Example#4

The magnitude of acceleration of centre of mass of the system is



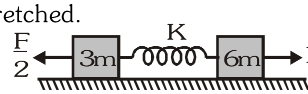
(A) 4 m/s^2 (B) 10 m/s^2 (C) 5 m/s^2 (D) 2 $\sqrt{2}$ m/s^2

Solution **Ans. (D)**

$$a = \frac{\text{Net force on system}}{\text{total mass of system}} = \frac{5g - \mu(5g)}{5+5} = \frac{50(1-0.2)}{10} = 4 \text{ m/s}^2 \text{ ; } a_{cm} = \left|\frac{m_1\vec{a}_1 + m_2\vec{a}_2}{m_1+m_2}\right| = \frac{a}{\sqrt{2}} = 2\sqrt{2} \text{ m/s}^2$$

Example#5

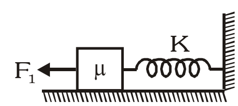
For shown situation find the maximum elongation in the spring. Neglect friction everywhere. Initially, the blocks are at rest and spring is unstretched.



(A) $\dfrac{4F}{3K}$ (B) $\dfrac{3F}{4K}$ (C) $\dfrac{4F}{K}$ (D) $\dfrac{2F}{K}$

Solution **Ans. (A)**

By using reduced mass concept this system can be reduced to



Where $\mu = \dfrac{(3m)(6m)}{3m+6m} = 2\text{ m}$ and F_1 = Force on either block w.r.t. centre of mass of the system

$$= \frac{F}{2} + (3m)a_{cm} = \frac{F}{2} + (3m)\left(\frac{F - F/2}{9m}\right) = \frac{F}{2} + \frac{F}{6} = \frac{2}{3}F$$

Now from work energy theorem , $\dfrac{2F}{3}x_m = \dfrac{1}{2}Kx_m^2 \Rightarrow x_m = \dfrac{4F}{3K}$

Example#6

A small sphere of mass 1kg is moving with a velocity $(6\hat{i} + \hat{j})$ m/s. It hits a fixed smooth wall and rebounds with velocity $(4\hat{i} + \hat{j})$ m/s . The coefficient of restitution between the sphere and the wall is-

(A) $\dfrac{3}{2}$ (B) $\dfrac{2}{3}$ (C) $\dfrac{9}{16}$ (D) $\dfrac{4}{9}$

Solution **Ans. (B)**

Impulse = Change in momentum $= 1(4\hat{i} + \hat{j}) - 1(6\hat{i} + \hat{j}) = -2\hat{i}$

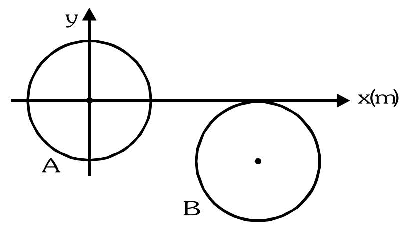
Which is perpendicular to the wall.

Component of initial velocity along $\hat{i} = 6\hat{i}$ $\Rightarrow$ Speed of approach = 6 m/s

Similarly speed of separation = $4ms^{-1}$ $\Rightarrow e = \dfrac{4}{6} = \dfrac{2}{3}$

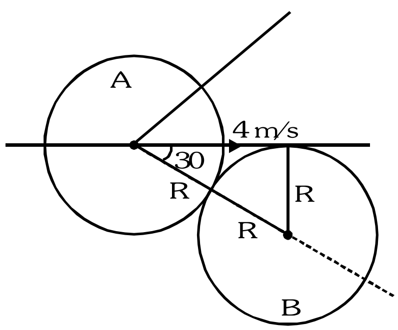
Example#7

Two smooth balls A and B, each of mass m and radius R, have their centre at (0, 0, R) and (5R, –R, R) respectively, in a coordinate system as shown. Ball A, moving along positive x-axis, collides with ball B. Just before the collision, speed of ball A is 4 m/s and ball B is stationary. The collision between the balls is elastic. Velocity of the ball A just after the collision is

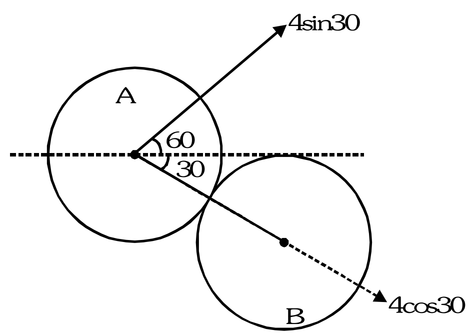


(a) $(\hat{i}+\sqrt{3}\hat{j})$ m/s (b) $(\hat{i}-\sqrt{3}\hat{j})$ m/s (c) $(2\hat{i}+\sqrt{3}\hat{j})$ m/s (d) $(2\hat{i}+2\hat{j})$ m/s

Solution **Ans. (A)**



Before collision

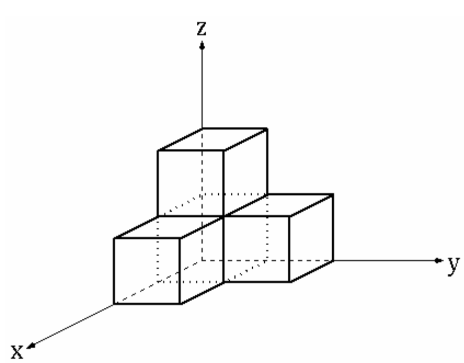


After collision

$$v_A = 4\sin 30°[\cos 60\hat{i} + \sin 60\hat{j}] = \hat{i} + \sqrt{3}\hat{j}$$

Example#8

Find the center of mass (x,y,z) of the following structure of four identical cubes if the length of each side of a cube is 1 unit.



(A) (1/2,1/2,1/2) (B)(1/3,1/3,1/3) (C) (3/4,3/4,3/4) (D)(1/2,3/4,1/2)

Solution **Ans. (C)**

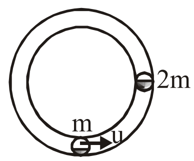
First we find the center of mass of each cube. It is located by symmetry: (0.5,0.5,0.5), (1.5,0.5,0.5), (0.5,1.5,0.5), (0.5,0.5,1.5) . Now we find the center of mass by treating the COM of each cube as a point particle:

$$x_{COM} = \frac{0.5+1.5+0.5+0.5}{4} = 0.75 \; ; \; y_{COM} = \frac{0.5+0.5+1.5+0.5}{4} = 0.75$$

$$z_{COM} = \frac{0.5+0.5+0.5+1.5}{4} = 0.75$$

Example#9

Two masses m and 2m are placed in fixed horizontal circular smooth hollow tube of radius r as shown. The mass m is moving with speed u and the mass 2m is stationary. After their first collision, the time elapsed for next collision. (coefficient of restitution e=1/2)



(A) $\frac{2\pi r}{u}$ (B) $\frac{4\pi r}{u}$ (C) $\frac{3\pi r}{u}$ (D) $\frac{12\pi r}{u}$

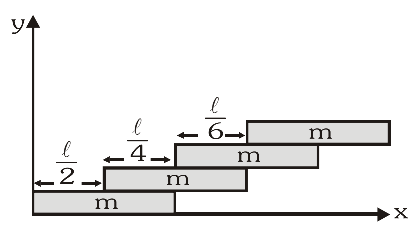
Solution **Ans. (B)**

Let the speeds of balls of mass m and 2m after collision be v_1 and v_2 as shown in figure. Applying conservation of momentum $mv_1 + 2mv_2 = mu$ & $-v_1 + v_2 = \frac{u}{2}$. Solving we get $v_1=0$ and $v_2 = \frac{u}{2}$

Hence the ball of mass m comes to rest and ball of mass 2m moves with speed $\frac{u}{2}$. $t = \frac{2\pi r}{u/2} = \frac{4\pi r}{u}$

Example#10

Find the x coordinate of the centre of mass of the bricks shown in figure :



(A) $\frac{24}{25}\ell$ (B) $\frac{25}{24}\ell$ (C) $\frac{15}{16}\ell$ (D) $\frac{16}{15}\ell$

Solution **Ans. (B)**

$$X_{cm} = \frac{m\left(\frac{\ell}{2}\right) + m\left(\frac{\ell}{2}+\frac{\ell}{2}\right) + m\left(\frac{\ell}{2}+\frac{\ell}{4}+\frac{\ell}{2}\right) + m\left(\frac{\ell}{2}+\frac{\ell}{4}+\frac{\ell}{6}+\frac{\ell}{2}\right)}{m+m+m+m} = \frac{25}{24}\ell$$

Example#11

Object A strikes the stationary object B with a certain given speed u head–on in an elastic collision. The mass of A is fixed, you may only choose the mass of B appropriately for following cases. Then after the collision :

(A) For B to have the greatest speed, choose $m_B = m_A$

(B) For B to have the greatest momentum, choose $m_B << m_A$

(C) For B to have the greatest speed, choose $m_B << m_A$

(D) For the maximum fraction of kinetic energy transfer, choose $m_B = m_A$

Solution **Ans. (B,C,D)**

$m_Au = m_Av_A + m_Bv_B$ and $e = 1 = \frac{v_B - v_A}{u} \Rightarrow v_B = \frac{2m_Au}{m_A + m_B}$

For $m_A >> m_B$, $v_B = 2u$

For $m_A = m_B$, $v_B = u$

For $m_A << m_B$, $v_B = 0$

kinetic energy $K_B = \frac{1}{2}m_Bv_B^2 = \frac{2m_Bu^2}{\left(1+\frac{m_B}{m_A}\right)^2}$

Example#12

A man is sitting in a boat floating in water of a pond. There are heavy stones placed in the boat.

(A) When the man throws the stones in water from the pond, the level of boat goes down.

(B) When the man throws the stones in water from the pond, the level of boat rises up.

(C) When the man drinks some water from the pond, the level of boat goes down

(D) When the man drinks some water from the pond, the level of boat remains unchanged.

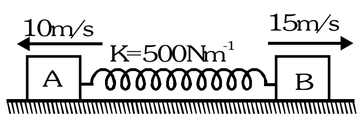
Solution **Ans. (B,D)**

For (A/B) : Force of buoyancy increases. Therefore level of boat rises up.

For (C/D): When man drinks some water, the level of boat remains unchanged.

Example#13

Two blocks A and B are joined together with a compressed spring. When the system is released, the two blocks appear to be moving with unequal speeds in the opposite directions as shown in figure.
Select incorrect statement(s) :



(A) The centre of mass of the system will remain stationary.

(B) Mass of block A is equal to mass of block B.

(C) The centre of mass of the system will move towards right.

(D) It is an impossible physical situation.

Solution **Ans. (BCD)**

As net force on system = 0 (after released)

So centre of mass of the system remains stationary.

Example#14

In which of the following cases, the centre of mass of a rod may be at its centre?

(A) The linear mass density continuously decreases from left to right.

(B) The linear mass density continuously increases from left to right.

(C) The linear mass density decreases from left to right upto centre and then increases.

(D) The linear mass density increases from left to right upto centre and then decreases.

Solution **Ans. (CD)**

Example#15

A man of mass 80 kg stands on a plank of mass 40 kg. The plank is lying on a smooth horizontal floor. Initially both are at rest. The man starts walking on the plank towards north and stops after moving a distance of 6 m on the plank. Then

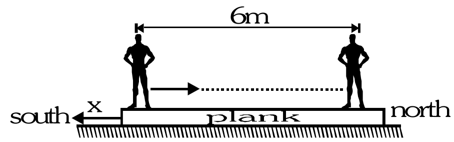
(A) The centre of mass of plank-man system remains stationary.

(B) The plank will slide to the north by a distance 4 m

(C) The plank will slide to the south by a distance 4 m

(D) The plank will slide to the south by a distance 12 m

Solution **Ans. (AC)**



Let x be the displacement of the plank.

Since CM of the system remains stationary

so $80(6-x) = 40x \Rightarrow 12 - 2x = x \Rightarrow x = 4m$

Example#16

A body moving towards a body of finite mass at rest, collides with it. It is impossible that

(A) both bodies come to rest

(B) both bodies move after collision

(C) the moving body stops and body at rest starts moving

(D) the stationary body remains stationary and the moving body rebounds

Solution **Ans. (AD)**

For (A) : Momentum can't destroyed by internal forces.

For (D) : If mass of stationary body is infinite then the moving body rebounds.

Example#17

Three interacting particles of masses 100 g, 200 g and 400 g each have a velocity of 20 m/s magnitude along the positive direction of x-axis, y-axis and z-axis. Due to force of interaction the third particle stops moving. The velocity of the second particle is $\left(10\hat{j}+5\hat{k}\right)$. What is the velocity of the first particle?

(A) $20\hat{i}+20\hat{j}+70\hat{k}$ (B) $10\hat{i}+20\hat{j}+8\hat{k}$ (C) $30\hat{i}+10\hat{j}+7\hat{k}$ (D) $15\hat{i}+5\hat{j}+60\hat{k}$

Solution **Ans. (A)**

Initial momentum = $m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 = 2\hat{i}+4\hat{j}+8\hat{k}$

When the third particle stops the final momentum = $m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 = 0.1\vec{v}_1 + 0.2\left(10\hat{j}+5\hat{k}\right)+\vec{0}$

By principle of conservation of momentum $0.1\,\vec{v}_1 + 2\hat{j}+\hat{k} = 2\hat{i}+4\hat{j}+8\hat{k}$; $\vec{v}_1 = 20\hat{i}+20\hat{j}+70\hat{k}$

Example#18 to 20

A bullet of mass m is fired with a velocity 10 m/s at angle θ with the horizontal. At the highest point of its trajectory, it collides head-on with a bob of mass 3m suspended by a massless string of length 2/5 m and gets embedded in the bob. After the collision the string moves through an angle of 60 .

18. The angle θ is

(A) 53 (B) 37 (C) 45 (D) 30

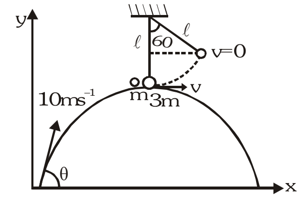
19. The vertical coordinate of the initial position of the bob w.r.t. the point of firing of the bullet is

(A) $\frac{9}{4}$ m (B) $\frac{9}{5}$ m (C) $\frac{24}{5}$ m (D) None of these

20. The horizontal coordinate of the initial position of the bob w.r.t. the point of firing of the bullet isn

(A) $\frac{9}{5}$ m (B) $\frac{24}{5}$ m (C) $\frac{9}{4}$ m (D) None of these

Solution



18. Ans. (B)

Velocity of combined mass just after collision

$m (10 \cos \theta) = 4mv \Rightarrow v = \frac{5}{2}\cos\theta$

But from energy conservation $\frac{1}{2}(4m)v^2 = 4mg\ell(1 - \cos 60)$

$\Rightarrow v = \sqrt{g\ell} = \frac{5}{2}\cos\theta \Rightarrow \cos\theta = \frac{2}{5}\sqrt{g\ell} = \frac{2}{5}\sqrt{10\times\frac{2}{5}} = \frac{4}{5} \Rightarrow \theta = 37$

19. Ans. (B)

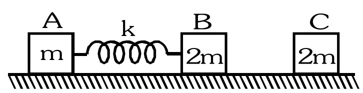
$$H_{max} = \frac{u^2\sin^2\theta}{2g} = \frac{(100)(9/25)}{20} = \frac{9}{5}m$$

20. Ans. (B)

$$\frac{R}{2} = \frac{2u^2\sin\theta\cos\theta}{2g} = \frac{(100)\left(\frac{3}{5}\right)\left(\frac{4}{5}\right)}{10} = \frac{24}{5}m$$

Example#21 to 23

Two blocks A and B of masses m and 2m respectively are connected by a spring of spring constant k. The masses are moving to the right with a uniform velocity v_0 each, the heavier mass leading the lighter one. The spring is of natural length during this motion. Block B collides head on with a third block C of mass 2m. at rest, the collision being completely inelastic.



21. The velocity of block B just after collision is-

(A) v_0 (B) $\frac{v_0}{2}$ (C) $\frac{3v_0}{5}$ (D) $\frac{2v_0}{5}$

22. The velocity of centre of mass of system of block A, B & C is-

(A) v_0 (B) $\frac{3v_0}{5}$ (C) $\frac{2v_0}{5}$ (D) $\frac{v_0}{2}$

23. The maximum compression of the spring after collision is -

(A) $\sqrt{\frac{mv_0^2}{12k}}$ (B) $\sqrt{\frac{mv_0^2}{5k}}$ (C) $\sqrt{\frac{mv_0^2}{10k}}$ (D) None of these

Solution

21. Ans. (B)

By applying conservation of linear momentum $2mv_0 = (2m + 2m)v \Rightarrow v = \frac{v_0}{2}$

22. Ans. (B)

$$v_{cm} = \frac{mv_0 + 2mv_0}{m + 2m + 2m} = \frac{3v_0}{5}$$

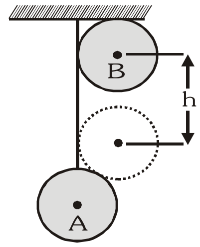
23. Ans. (B)

At maximum compression, velocity of all blocks are same & equal to velocity of centre of mass.

$$\frac{1}{2}kx_m^2 = \left[\frac{1}{2}mv_0^2 + \frac{1}{2}(4m)\left(\frac{v_0}{2}\right)^2\right] - \frac{1}{2}(5m)\left(\frac{3v_0}{5}\right)^2 \Rightarrow \frac{1}{2}kx_m^2 = \frac{1}{10}mv_0^2 \Rightarrow x_m = \sqrt{\frac{mv_0^2}{5k}}$$

Example#24

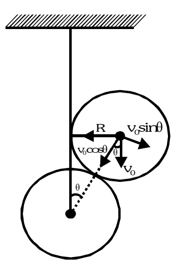
A smooth ball A of mass m is attached to one end of a light inextensible string, and is suspended from fixed point O. Another identical ball B, is dropped from a height h, so that the string just touches the surface of the sphere.



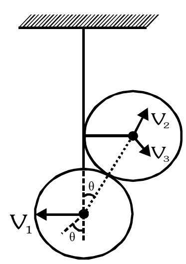
	Column I		Column II
(A)	If collision between balls is completely elastic then speed of ball A just after collision is	(P)	$\frac{3m}{5}\sqrt{2gh}$
(B)	If collision between balls is completely elastic then impulsive tension provided by string is	(Q)	$\frac{\sqrt{6gh}}{5}$
(C)	If collision between balls is completely inelastic then speed of ball A just after collision is	(R)	$\frac{6m}{5}\sqrt{2gh}$
(D)	If collision between balls is completely inelastic then impulsive tension provided by string is	(S)	$\frac{2\sqrt{6gh}}{5}$
		(T)	None of these

Solution **Ans. (A) → (S), (B) → (R), (C) → (Q), (D) → (P)**

For(A) $v_0 = \sqrt{2gh}$, $\sin\theta = \frac{R}{2R} = \frac{1}{2}$. By definition of e, e = 1 = $\frac{v_1 \sin\theta + v_2}{v_0 \cos\theta}$



Just Before Collision



Just after Collision

Let impulse given by ball B be N. then by impulse momentum theorem
$N = m(v_2 + v_0\cos\theta)$ & $N\sin\theta = mv_1$

$$\Rightarrow v_1 = \frac{2v_0 \sin\theta\cos\theta}{1+\sin^2\theta} = \frac{\left(2\sqrt{2gh}\right)\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}{1+\left(\frac{1}{2}\right)^2} = \frac{2\sqrt{6gh}}{5}$$

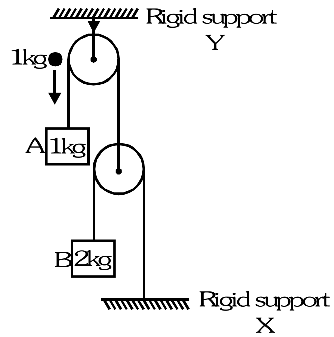
For(B) Impulsive tension = N cos θ = $\left(\frac{mv_1}{\sin\theta}\right)$ cos θ = mv_1 cotθ$=\frac{6m}{5}\sqrt{2gh}$

For (C) For completely inelastic collision e=0, so $v_1 \sin\theta + v_2 = 0 \Rightarrow v_1 = \frac{v_0 \sin\theta\cos\theta}{1+\sin^2\theta} = \frac{\sqrt{6gh}}{5}$

For (D) Impulsive tension = Ncosθ$=\left(\frac{mv_1}{\sin\theta}\right)$ cosθ = mv_1 cotθ = $\frac{3m}{5}\sqrt{2gh}$

Example#25

Collision between ball and block A is perfectly inelastic as shown. If impulse on ball (at the time of collision) is J then



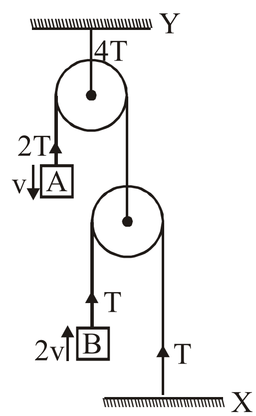
	Column- I		Column-II
(A)	Net impulse on block A is	(P)	J
(B)	Net impulse on block B is	(Q)	4J/9
(C)	Impulse due to rigid support Y is	(R)	16J/9
(D)	Impulse due to rigid support X is	(S)	2J/9
		(T)	J/9

Solution (A) T (B) Q (C) R (D) Q

By using impulse momentum theorem :

on A : J–2T = 1(v)

on B : T = 2(2v) Therefore J = 9 v



Net impulse on A = 1(v) = $\frac{J}{9}$

Net impulse on B = $4v = \frac{4J}{9}$

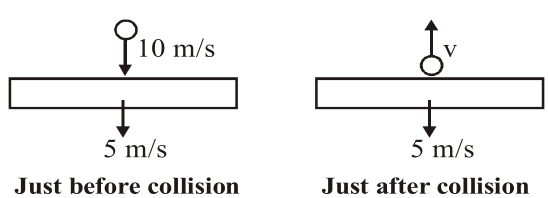
Impulse due to rigid support Y = 4T = $\frac{16J}{9}$

Impulse due to rigid support X = T = $\frac{4J}{9}$

Example#26

A ball moving vertically downward with a speed of 10 m/s collides with a platform. The platform moves with a velocity of 5 m/s in downward direction. If e = 0.8, find the speed (in m/s) of the ball just after collision.

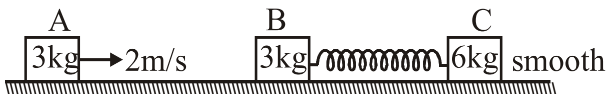
Solution Ans. 1



By definition of e : $e = \frac{v_2 - v_1}{u_1 - u_2}$; we have $0.8 = \frac{v+5}{10-5}$ $\Rightarrow$ v = 1 m/s

Example#27

For shown situation, if collision between block A and B is perfectly elastic, then find the maximum energy stored in spring in joules.



Solution **Ans. 4**

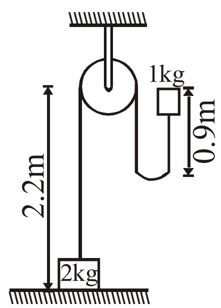
At maximum compression of spring, velocities of block B and C are same (say v_0)

then by conservation of linear momentum $3(2) = (3+6)v_0 \Rightarrow v_0 = \frac{2}{3}$ m/s

At this instant energy stored in spring $= \frac{1}{2}(3)(2)^2 - \frac{1}{2}(3+6)\left(\frac{2}{3}\right)^2 = 6 - 2 = 4$ J

Example#28

In the shown figure, the heavy block of mass 2 kg rests on the horizontal surface and the lighter block of mass 1 kg is dropped from a height of 0.9 m. At the instant the string gets taut, find the upward speed (in m/s) of the heavy block.



Solution **Ans. 2**

Velocity of lighter block at the instant the string just gets taut $v = \sqrt{2gh} = \sqrt{2 \times 10 \times 1.8} = 6$ m/s

Now by impulse - momentum theorem, let common speed be v_1 then $(2+1)\, v_1 = (1)\, v \Rightarrow v_1 = \frac{v}{3} = \frac{6}{3} = 2$ m/s

Example#29

Two balls of equal mass have a head-on collision with speed 6 m/s. If the coefficient of restitution is $\frac{1}{3}$, find the speed of each ball after impact in m/s.

Solution **Ans. 2**

Just before collision

Just after collision

By definition of e : $e = \frac{v_2 - v_1}{u_1 - u_2} \Rightarrow \frac{1}{3} = \frac{v+v}{6+6} \Rightarrow v = 2$ m/s

Example#30

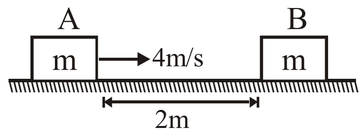
A thin rod of length 6 m is lying along the x-axis with its ends at x=0 and x = 6 m. Its linear density (mass/length) varies with x as kx^4. Find the position of centre of mass of rod in meters.

Solution **Ans. 5**

$$x_{cm} = \frac{\int x\,dm}{\int dm} = \frac{\int_0^6 x\left(kx^4 dx\right)}{\int_0^6 \left(kx^4 dx\right)} = \frac{\int_0^6 x^5 dx}{\int_0^6 x^4 dx} = \frac{\left(\frac{x^6}{6}\right)_0^6}{\left(\frac{x^5}{5}\right)_0^6} = 5\text{m}$$

Example#31

The friction coefficient between the horizontal surface and blocks A and B are $\frac{1}{15}$ and $\frac{2}{15}$ respectively. The collision between the blocks is perfectly elastic. Find the separation (in meters) between the two blocks when they come to rest.



Solution **Ans. 5**

Velocity of block A just before collision $v_A = \sqrt{u_A^2 - 2\mu g x} = \sqrt{16 - 2\left(\frac{1}{15}\right)(10)(2)} = \sqrt{\frac{40}{3}}$

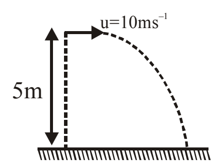
Velocity of Block B just after collision $v_B = v_A = \sqrt{\frac{40}{3}}$

Velocity of Block A just after collision = 0

Total distance travelled by block B = $\frac{v_B^2}{2\mu g} = \frac{40/3}{2\left(\frac{2}{15}\right)(10)} = 5\text{m}$

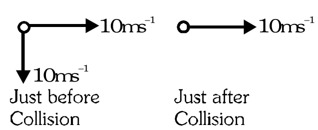
Example#32

A ball of mass 1 kg is projected horizontally as shown in figure. Assume that collision between the ball and ground is totally inelastic. The kinetic energy of ball (in joules) just after collision is found to be 10α. Find the value of α.



Solution **Ans. 5**

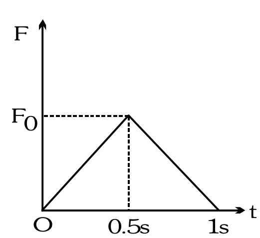
Vertical velocity just before collision $v_y = \sqrt{2gh} = \sqrt{2 \times 10 \times 5} = 10$ m/s



$\Rightarrow$ Kinetic energy of ball just after collision = $\frac{1}{2}$ 1 10^2 = 50 J

Example#33

A body of mass 1 kg moving with velocity 1 m/s makes an elastic one dimensional collision with an identical stationary body. They are in contact for brief time 1 sec. Their force of interaction increases from zero to F_0 linearly in time 0.5 s and decreases linearly to zero in further time 0.5 sec as shown in figure. Find the magnitude of force F_0 in newton.



Solution **Ans. 2**

In the one dimensional elastic collision with one body at rest, the body moving initially comes to rest & the one which was at rest earlier starts moving with the velocity that first body had before collision.

so, if m & V_0 be the mass & velocity of body,

the change in momentum = $mV_0 \Rightarrow \int Fdt = mV_0 \Rightarrow \int Fdt = mV_0 \Rightarrow F = \frac{2mV_0}{\Delta t} = 2N$

Example#34

An object A of mass 1 kg is projected vertically upward with a speed of 20 m/s. At the same moment another object B of mass 3 kg, which is initially above the object A, is dropped from a height h = 20 m. The two point like objects (A and B) collide and stick to each other. The kinetic energy is K (in J) of the combined mass just after collision, find the value of K/25.

Solution **Ans. 2**

Using relative motion, the time of collision is $t = \frac{h}{20+0} = 1s$

By conservation of momentum for collision 3(10) + 1 (–10) = 4 (V) $\Rightarrow$V = 5 m/s

$KE = \frac{1}{2}(4)(5)^2 = 50J$

Example#35

An 80 kg man is riding on a 40 kg cart travelling at a speed of 2.5 m/s on a frictionless horizontal plane. He jumps off the cart, such that, his velocity just after jump is zero with respect to ground. The work done by him on the system during his jump is given as $\frac{A}{4}$ KJ (A$\in$ integer). Find the value of A.

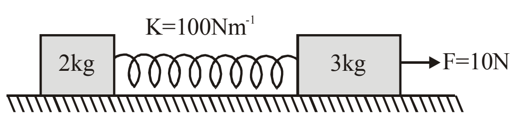
Solution **Ans. 3**

By conservation of linear momentum (80 + 40) (2.5) = 80 (0) + 40 (v) $\Rightarrow$ v = 7.5 m/s

work done = $\Delta KE = \frac{1}{2}40(7.5)^2 - \frac{1}{2}(80+40)(2.5)^2 = 750$ J

Example#36

At t=0, a constant force is applied on 3 kg block. Find out maximum elongation in spring in cm.



Solution **Ans. 8**

Given system can be reduced by using reduced mass concept $\mu = \frac{2\times 3}{2+3} = \frac{6}{5}kg$

and $F_{reduced}$ = force on any block w.r.t. centre of mass = $\frac{m_1F_2 + m_2F_1}{m_1+m_2} = \frac{2\times 10 + 3\times 0}{2+3} = 4$ N

$\frac{1}{2}kx^2 = 4x \Rightarrow x = \frac{8}{k} = \frac{8}{100} = 8\times 10^{-2}m = 8$ cm